ON THE EQUATION $dy/dx=f(x, y)$[1]

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We consider here the differential equation $dy/dx=f(x, y)$, where $f(x, y)$ is a one-valued function defined on an open region R of the xy-plane. By a solution curve of this equation we mean a curve $y=y(x)$ which has a derivative at every point and which satisfies everywhere the differential equation. There are known sufficiency conditions on f for the existence of a one parameter family of solution curves simply covering R. But, as Professor Bamforth mentioned to me orally, there seems to be in the literature no corresponding necessary conditions. We shall prove that one necessary condition is that f be the limit of a sequence of continuous functions.[2]

A curve is the xy-plane will be termed a *continuous function curve* if the points of the curve are the points $(x, \phi(x))$, $a<x<b$, where $\phi(x)$ is a one-valued continuous function defined on an open interval (a, b). An open region R of the xy-plane will be said to be simply covered by a set $\mathcal{J}$ of such curves if:

(1) Every point of R is on one and only one curve of $\mathcal{J}$.

(2) Every curve of $\mathcal{J}$ stretches from boundary to boundary of R; that is, if S is any set of points on a curve C of $\mathcal{J}$, each limit point of S is either itself a point of C or a boundary point of R.

THEOREM. *If an open region R of the xy-plane is simply covered by a set $\mathcal{J}$ of continuous function curves $y=\phi(x)$, then for every point (x_0, y_0) of R there exists an open subregion R_0 of R containing (x_0, y_0) such that the family of curves constituted by the portions of the curves of $\mathcal{J}$ in R_0 is representable by the equation $y=\phi(x, a)$, where ϕ is a continuous function of x and the parameter a.*

PROOF. Let (x_0, y_0) be a point of such a given region R; R_1 a rectangle interior to R with (x_0, y_0) as center; and h a positive number such that the points (x_0, y_0-h) and (x_0, y_0+h) are inside R_1. Since $\mathcal{J}$ simply covers R, there exist curves $y=\phi_1(x)$ and $y=\phi_2(x)$ of $\mathcal{J}$ containing the points (x_0, y_0-h) and (x_0, y_0+h) respectively. Also, the continuity of $\phi_1(x)$ and $\phi_2(x)$ insures the existence of an open interval I containing x_0 such that the points of the curves $y=\phi_1(x)$ and

[1] I wish to express my thanks to Professor Henry Blumberg for his aid in the preparation of this paper.

[2] Of course, as is well known, the derivative $f'(x)$ of a function $f(x)$ has this property. It may be expected that $f(x, y)$ necessarily has also other properties corresponding to known properties of $f'(x)$.