MEASURABILITY AND MODULARITY IN THE THEORY OF LATTICES[1]

M. F. SMILEY

In a previous note[2] a notion of measurability (with respect to a function μ) of elements of an arbitrary lattice was introduced. Our purpose there was to study closure properties of the subset of measurable elements. To do this it was convenient to assume that the lattice was modular. Results of V. Glivenko,[3] and of L. R. Wilcox and the author[4] indicate that the idea of measurability and that of modularity are intimately related. The purpose of this note is to exhibit a further relationship which does not depend on metric properties of the function μ.

In a lattice L we call the ordered pair (a, b) of elements of L *modular* and write $(a, b)M$ in case $(a_1+a)b=a_1+ab$ for every $a_1 \leqq b$. This relation has been studied by L. R. Wilcox.[5] If $\mu(a)$ is a real valued function defined on L we say that an element $c \in L$ is μ-measurable in case

$$\mu(c) + \mu(b) = \mu(c + b) + \mu(cb)$$

for every $b \in L$. The symbol $L(\mu)$ will denote the totality of elements of L which are μ-measurable. We call μ *proper* in case $a \leqq b$ with $\mu(a)=\mu(b)$ implies $a=b$.

Before we discuss the general case, let us consider a lattice L_0 of finite dimension (that is, satisfying both chain conditions) in which every principal chain joining two elements a, $b \in L_0$ has the same length $n(a, b)$. Examples of such lattices are well known.[6] Let 0 de-

[1] Presented to the Society, February 24, 1940.

[2] *A note on measure functions in a lattice*, this Bulletin, vol. 46 (1940), pp. 239–241. This concept specializes to that of permutability with all subgroups (that is, the *quasi-normality* of O. Ore, *Structures and group theory* I, Duke Mathematical Journal, vol. 3 (1937), p. 162) if we set $\mu(s)=\log o(s)$ in the lattice of subgroups of a finite group, and to that of *measurability* in the sense of Carathéodory with respect to an *outer measure* function.

[3] *Contributions à l'étude des systèmes de choses normées*, American Journal of Mathematics, vol. 59 (1937), pp. 933–934.

[4] *Metric lattices*, Annals of Mathematics, (2), vol. 40 (1939), p. 313.

[5] *Modularity in the theory of lattices*, Annals of Mathematics, (2), vol. 40 (1939), p. 491 ff.

[6] The "exchange" lattices discussed by Saunders Mac Lane (*A lattice formulation for transcendence degrees and p-bases*, Duke Mathematical Journl, vol. 4 (1938), pp. 455–468) as well as the "semi-modular" lattices of finite dimension of Wilcox (op. cit., pp. 502–505) have these properties. We make no use, however, of the metric properties of these systems.