RATIONAL METHODS IN MATRIX EQUATIONS[1]

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I. **Introduction.**[2] I shall start with what I hope will not prove an overelaborate statement of the limitations of this paper in scope and treatment. I shall assume throughout a knowledge of the definitions of a field, a division algebra, of matrices and rational operations thereon. I shall also need to assume a knowledge of what is meant by the invariant factors and elementary divisors of a matrix. Little essential will be lost if the only field considered by the listeners is the rational number system and the only division algebra that of quaternions over the rational field.

Consider a field $\mathfrak{K}$ and a system of constant matrices $A_1, \cdots, A_l$, unknown matrices $X_1, \cdots, X_n$ and equations

$$\phi_i(A_1, \cdots, A_l, X_1, \cdots, X_n) = 0 \tag{1}$$

where the ϕ_i's are polynomials with coefficients in $\mathfrak{K}$. If the elements of A_i are a_{ijk} and of X_i are x_{ijk}, (1) is equivalent to a system

$$\psi_s(\cdots, x_{ijk}, \cdots) = 0 \tag{2}$$

where the ψ_s are polynomials with coefficients in $\mathfrak{K}$. (If $\mathfrak{K}$ is replaced by a division algebra $\mathfrak{d}$ over $\mathfrak{K}$, the number of equations in (2) is merely enlarged.) We have therefore "reduced" the equations (1) to those of (2). This process we shall technically designate without great exaggeration as the "worst possible algorithm," or, following modern style, W.P.A. This indicates that no "tour de force" which shows that ultimately a matrix problem can be solved in a finite time, but shows little else, is of interest. This is a topic in which the above simple proof of the existence of inelegant methods means that we need only pay attention to results that essentially use the matricial properties of matrices, only to results and methods having at least a minimum degree of elegance.

Partly as a consequence of the above, this lecture, as is often the case, is not so much a description of broad theories of the nature we desire, as a report on what special cases have been found to be seduci-

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