ON UNCONDITIONAL CONVERGENCE IN NORMED VECTOR SPACES

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Suppose X is a complete normed vector or Banach space of elements x. Orlicz[1] has given the following two definitions of unconditional convergence of an infinite series $\sum_n x_n$ of elements from X and proved their equivalence:

A. $\sum_n x_n$ is unconditionally convergent if and only if any rearrangement of the series is convergent.

B. $\sum x_n$ is unconditionally convergent if and only if $\sum_k x_{n_k}$ converges, where $\{x_{n_k}\}$ is any subsequence of $\{x_n\}$.

Pettis[2] has shown that either of these conditions is equivalent to the statement:

C. Every subseries of $\sum_n x_n$ is weakly convergent to an element of X, that is, $\{x_{n_k}\}$ implies the existence of an element x_σ such that, for every $\bar{x}$ of the conjugate space $\overline{X}$, $\sum_k \bar{x}(x_{n_k}) = \bar{x}(x_\sigma)$.

In proving this equivalence, he shows that condition C implies the following:

D. $\lim_n \sum_{m=n}^{\infty} \bar{x}(x_m) = 0$ uniformly for $\|\bar{x}\| = 1$.

E. H. Moore[3] has shown that for real, complex or quaternionic numbers, absolute and therefore unconditional convergence is equivalent to the following definition of convergence:

Let σ be any finite subset $n_1, \cdots, n_k$ of the positive integers, and denote $\sum_{i=1}^{k} x_{n_i}$ by $\sum_\sigma x_n$. Then

E. $\sum x_n$ converges in the σ-sense, if $\lim_\sigma \sum_\sigma x_n$ exists, where the limit is the Moore-Smith limit, and $\sigma_1 \geqq \sigma_2$ means that σ_1 contains all of the numbers in σ_2.[4]

Obviously the Moore-Smith limit can be extended to normed vector spaces, and the fundamental properties carry over. It is the purpose of this note to show that convergence in the σ-sense is equivalent to each of the conditions A, B, and D, that is, A, B, D and E are equivalent definitions of unconditional convergence, to which the

[1] *Ueber unbedingte Konvergenz in Funktionenräumen*, Studia Mathematica, vol. 4 (1933), pp. 33–38.

[2] *Integration in vector spaces*, Transactions of this Society, vol. 44 (1938), pp. 281–282.

[3] *General Analysis*, Memoirs of the American Philosophical Society, vol. 1, part 2, 1939, p. 63.

[4] See Alaoglu, Annals of Mathematics, (2), vol. 41 (1940), p. 259, where a similar definition for weak unconditional convergence is given.