NOTE ON SOME ELEMENTARY PROPERTIES OF POLYNOMIALS

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In a previous paper T. Grünwald[1] and I proved that if $f(x)$ is a polynomial of degree $n \geqq 2$ and satisfies the following conditions:

$$\text{(1)} \qquad \begin{aligned} &\text{all roots of } f(x) \text{ are real}, \quad f(-1) = f(+1) = 0, \\ &f(x) \neq 0 \quad \text{for} \quad -1 < x < 1, \quad \max_{-1<x<1} f(x) = 1, \end{aligned}$$

then

$$\text{(2)} \qquad \int_{-1}^{+1} f(x) \leqq \frac{4}{3}.$$

Equality occurs only for $f(x) = 1 - x^2$.

This result can be generalized as follows: Suppose $f(x)$ satisfies (1) and let $f(a) = f(b) = d \leqq 1$, $-1 < a < b < 1$; then

$$\text{(3)} \qquad b - a \leqq 2(1 - d)^{1/2}.$$

Again equality occurs only for $f(x) = 1 - x^2$. It is clear that (2) follows from (3) by integration with respect to d.

PROOF. Instead of (3) we prove the following slightly more general result: Let $f(x)$ satisfy (1), and determine the greatest positive constant c_f such that

$$f(a)f(a + c_f) = d^2, \quad -1 < a < a + c_f < 1;$$

then

$$\text{(4)} \qquad c_f \leqq 2(1 - d)^{1/2}.$$

Equality holds only for $f(x) = 1 - x^2$, $a = -(1 - d)^{1/2}$.

Suppose there exists a polynomial of degree $n > 2$ satisfying (1) with $c_f \geqq 2(1 - d)^{1/2}$; then we will prove that there exists a polynomial of degree $n - 1$ with $c_f > 2(1 - d)^{1/2}$; and this proves (4) since it is easy to prove that (4) is satisfied for polynomials of second degree, that is, for $1 - x^2$.

Denote the roots of $f(x)$ by $x_1 = -1$, $x_2 = 1$, $x_3, \cdots, x_n$ and suppose first that for $i > 2$ the x_i are not all of the same sign. Let x_n be the largest positive root and x_{n-1} the smallest negative root, and denote by y the root of $f'(x)$ in $(-1, +1)$. Consider the polynomial of degree n

[1] Annals of Mathematics, (2), vol. 40 (1939), pp. 537–548.