A CHARACTERIZATION OF EUCLIDEAN SPACES

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The purpose of this paper is to give an elementary proof of the fact that a Banach space in which there exist projection transformations of norm one on every two-dimensional linear subspace is a euclidean space. S. Kakutani [1] has pointed out that a modification of a proof due to Blaschke [2] will prove this theorem. F. Bohnenblust has been able to establish this theorem for the complex case by still another method.[1]

A Banach space is a linear, normed, complete space [3, chap. 5]. A euclidean space of dimension α, where α is any cardinal number, is defined to be the Banach space of sequences x_ν of real numbers where ν ranges over a class of cardinal number α, and $\sum x_\nu^2$ is finite and equal to the square of the norm [4]. We consider only spaces having at least three linearly independent elements.

P. Jordan and J. von Neumann have shown [5] that a Banach space which is euclidean in every two-dimensional linear subspace is itself a euclidean space. It is thus sufficient to show that the "unit sphere" S for any three-dimensional linear subspace is an ellipsoid.

Because of the norm properties, S is a convex body symmetric about the origin o, and contains o as an interior point. Let γ be a plane containing o and let C_γ be the curve of intersection of γ and the boundary S' of S. The existence for each γ of a projection operation of norm one, whose direction of projection is that of the unit vector v_γ, implies that the cylinder generated by lines of direction v_γ tracing C_γ contains S. Our theorem is therefore an immediate consequence of the following lemma on convex bodies (which need not be symmetric about o).

LEMMA.[2] *If S is a convex body such that for every γ there exists a cylinder generated by C_γ containing S, then S is an ellipsoid.*

We topologize the planes γ by representing each by its direction cosines as a point on the unit sphere and using the usual topology of the unit sphere.

The proof of the lemma is divided into two parts. We first show that v_γ is uniquely determined by γ, that v_γ is a continuous function of γ, and that S' has a tangent plane at each of its points. It is then

[1] F. Bohnenblust's result is not yet published.

[2] W. Blaschke has proved a similar theorem under the assumption that there exists a tangent plane at each point of S' [2].