A NOTE ON MAXIMUM MODULUS AND THE ZEROS OF AN INTEGRAL FUNCTION

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Let $f(z)$ be an integral function of finite order $\rho \geqq 0$, let $M(r, f) = M(r) = \max_{|z|=r} |f(z)|$, and let $n(r, f) = n(r)$ be the number of zeros of $f(z)$ in $|z| \leqq r$ and on its circumference. I have discussed elsewhere[1] the behaviour of $g(r) = \log M(r)/n(r)$, as $r \to \infty$, and have proved that *for every canonical product function* $f(z)$

$$\liminf_{r=\infty} \frac{\log M(r)}{n(r)\phi(r)} = 0, \tag{1}$$

where $\phi(r)$ *is any positive L function*[2] *such that*

$$\int^{\infty} \frac{dx}{x\phi(x)} < A = \text{const.} \tag{2}$$

The question arises how large and how small $g(r)$ and $G(r) = T(r, f)/n(r)$ can be, where $T(r, f) = T(r)$ is the Nevanlinna characteristic function for $f(z)$. I prove in this note the following result.

THEOREM. *Given* $\rho \geqq 0$ *and* $\psi(x)$ *any positive function such that*

$$\limsup_{x=\infty} \frac{\log \psi(x)}{\log x} \leqq \rho.$$

There exists an integral function $F(z)$ *of order* ρ *for which*

$$\limsup_{r=\infty} \frac{T(r, F)}{\psi(r)n(r, F)} = \infty \tag{3}$$

and an integral function $f(z)$ *of order* ρ *for which*

$$\liminf_{r=\infty} \frac{\psi(r)T(r, f)}{n(r, f)} = 0. \tag{4}$$

PROOF. We shall first construct an integral function $f(z)$ of order ρ for which

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[1] (i) *A theorem on integral functions of integral order*, Journal of the London Mathematical Society, vol. 15 (1940), pp. 23–31. (ii) *On integral functions of integral or zero order*, to be published. (iii) *On integral functions of perfectly regular growth*, Journal of the London Mathematical Society, vol. 14 (1939), pp. 293–302.

[2] For definition see G. H. Hardy, *Orders of Infinity*, 1924, p. 17.