THE RADIUS AND MODULUS OF n-VALENCE FOR ANALYTIC FUNCTIONS WHOSE FIRST $n-1$ DERIVATIVES VANISH AT A POINT

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The principal result of this note is the determination of the precise radius and modulus of n-valence for the class of functions $f(z)=z^n+a_{n+1}z^{n+1}+\cdots$ analytic and less than or equal to M in modulus in $|z|\leqq 1$. This result readily leads to the radius and modulus of n-valence for the more general class of functions $f(z)=az^n+a_{n+1}z^{n+1}+\cdots$ analytic and less than or equal to M in modulus in $|z|\leqq R$. Finally, we note certain approximations which rather naturally suggest themselves in a search for more easily calculable constants.

We consider only expansions about the origin of functions $f(z)$ with $f(0)=0$, the generalization to expansions about a of functions $f(z)$ with $f(a)=b$ being obvious. Each circle mentioned will be understood to have the origin ($w=0$ or $z=0$) as center. The phrases *radius of n-valence* and *modulus of n-valence*, which usually refer to a class of functions, will also be used with reference to a single function. The radius of n-valence of the function $f(z)$ is the radius of the largest circle within which $f(z)$ assumes no value more than n times, and assumes at least one value n times. The modulus of n-valence of $f(z)$ is the radius of the largest circle of which the interior is covered exactly n times by the map under $f(z)$ of $|z|<\rho$, where ρ is the above radius of n-valence. Consider now one of the classes defined above. It is obvious that for each function $w=f(z)$ of the class there is a neighborhood of $z=0$ in which the function assumes no value more than n times, and assumes exactly n times every value in a sufficiently small neighborhood of $w=0$. The radius of n-valence ρ_n of the class is the radius of the largest circle within which *no* function of the class assumes a value more than n times. The modulus of n-valence m_n of the class is the radius of the largest circle of which the interior is covered exactly n times by the map of $|z|<\rho_n$ under *every* function of the class.

THEOREM. *Consider the class of functions* $f(z)=z^n+a_{n+1}z^{n+1}+\cdots$ *analytic and less than or equal to* M $(M>1)$[1] *in modulus in* $|z|\leqq 1$,

[1] The restriction to $M>1$ is necessary. By the Cauchy coefficient inequality, $M\geqq 1$, and if $M=1$ the class consists of the single function $f(z)=z^n$ for which the theorem is false.