NULLIFYING FUNCTIONS[1]

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Introduction. A function $f(x)$ defined on the unit interval $(0, 1)$ will be called nullifying if we can find a set S of $(0, 1)$ for which $m(S)=1$, $m(\{f(x); x \varepsilon S\})=0$. Examples of homeomorphisms which are nullifying and hence termed singular are well known.[2] We shall however consider simply the nullifying property itself.

If $f(x)$ is nullifying and $\phi(x)$ is not, one might expect that $\phi(x)+f(x)$ shares with ϕ the property of being not nullifying. But this is not always true as the following example shows. Let $.\alpha_1\alpha_2 \cdots$ denote the dyadic expansion for x, that is, $x=\alpha_1/2+\alpha_2/2^2+\cdots$ with $\alpha_i=0$ or 1. Let $v_1(x)=.\alpha_1 0\alpha_3 0 \cdots$ and $v_2(x)=.0\alpha_2 0\alpha_4 \cdots$. It is easily verified that both v_1 and v_2 are nullifying. Hence $f(x)=1-v_1$ is also nullifying. Let $\phi=x$. Then $f+\phi=1-v_1+x=1+v_2$ is also nullifying.

But this suggests the question: Does there exist a nullifying function $f(x)$ such that $f(x)+\rho x$ is nullifying for every value of ρ? We construct such a function in the present note.

Our method of proof can be summarized as follows. Considering the set $\{f(x)+\rho x; x \varepsilon (0, 1)\}$, we let $\rho=\cot\theta$. (Note $\theta\neq 0$.) If this set has measure zero, this will still be true if we multiply by $\sin\theta$ and conversely. Thus we may consider the sets $\{x\cos\theta+f(x)\sin\theta; x \varepsilon (0, 1)\}$ for each $\theta\neq 0$ between $-\pi/2$ and $\pi/2$. If we consider the line through the origin of inclination θ, we can assign a coordinate to each of its points in the usual manner with positive direction to the right or, in the case of the y axis, upwards. The set $\{x\cos\theta+f(x)\sin\theta; x \varepsilon (0, 1)\}$ is the set of coordinates of the projection onto this line of the graph of $f(x)$. Thus it suffices to find a function $f(x)$ which is such that the projection of its graph onto any line not parallel to the x axis is of measure zero. We proceed to find the graph of such a function by an intersection process on sets in the plane. This process is described in detail in what follows.

A more general question is: Given $F(x, y, \rho)$, under what circumstances can we find a function $f(x)$ such that $F(x, f(x), \rho)$ is nullifying in x for every value of ρ? It is comparatively easy to abstract the properties of $F=y+\rho x$ which are essential to the present discussion, and these will prove sufficient to obtain an answer to the question.

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[2] Cf. E. R. van Kampen and Aurel Wintner, *On a singular monotone function*, Journal of the London Mathematical Society, vol. 12 (1937), pp. 243–244. References to preceding examples are given in this paper.