DIOPHANTINE EQUATIONS OF DEGREE[1] n

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In a recent issue of the National Mathematics Magazine,[2] W. V. Parker and the author obtained solutions of the Diophantine equation $F(x_1, \cdots, x_p) = G(y_1, \cdots, y_q)$, where F and G are homogeneous polynomials, with integral coefficients, of degree 3, and F is such that for a set of integers $x_i = a_i$ (not all zero), $\partial F/\partial x_i = 0$, $(i = 1, \cdots, p)$. In this paper the above is extended to functions of degree n. One type, which satisfies the conditions of the main theorem, is also solved by an entirely different method. The solutions obtained are in terms of arbitrary parameters, and they are integral for an integral choice of the parameters.

If $x_i = \alpha_i$, $y_k = \beta_k$ is a solution of the equation $f(x_1, \cdots, x_p) = g(y_1, \cdots, y_q)$, where f and g are homogeneous polynomials, with integral coefficients, of degrees n and m respectively, and there are no integers $s > 1$, α_i', β_k' such that $\alpha_i = s^\lambda \alpha_i'$, $\beta_k = s^\mu \beta_k'$ where λ, μ are relatively prime positive integers such that $\lambda n = \mu m$, then $x_i = \alpha_i$, $y_k = \beta_k$ is said to be a primitive solution. If $x_i = \alpha_i$, $y_k = \beta_k$ is a primitive solution of the above equation, then $x_i = \alpha_i t^\lambda$, $y_k = \beta_k t^\mu$ (derived from this primitive solution), where λ, μ are any positive integers such that $\lambda n = \mu m$, is also a solution. Two solutions are said to be equivalent if they are derived from the same primitive solution.

THEOREM 1. *Let $f(x_1, \cdots, x_p)$, $g(y_1, \cdots, y_q)$ be homogeneous polynomials with integral coefficients, of degrees n and m respectively. Let $a_1, \cdots, a_p$ be integers not all zero such that the partial derivatives of f of all orders less than $n-1$ vanish*[3] *when $x_i = a_i$. Then every solution in integers x_i, y_k of*

$$f(x_1, \cdots, x_p) = g(y_1, \cdots, y_q), \tag{1}$$

for which

$$\sum_{j=1}^{p} a_j \frac{\partial f}{\partial x_j} \neq 0, \tag{2}$$

is equivalent to one of the infinitude of solutions given by

$$x_i = a_i s t^{\lambda-1} + \alpha_i t^\lambda, \quad y_k = \beta_k t^\mu, \quad i = 1, 2, \cdots, p;\ k = 1, 2, \cdots, q, \tag{3}$$

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[1] Presented to the Society, December 2, 1939.

[2] *On cubic Diophantine equations*, vol. 13 (1938), pp. 115–117.

[3] It follows from Euler's theorem that the function itself vanishes for this choice of x_i.