AN ALMOST UNIVERSAL FORM

GORDON PALL

P. R. Halmos[1] obtained the 88 possible forms (a, b, c, d), $0<a\leqq b\leqq c\leqq d$, which represent all positive integers with one exception, and proved that property for all except for the form $h=(1, 2, 7, 13)$. A proof for h follows.

The forms $f=(1, 2, 7)$ and $g=(1, 1, 14)$ constitute the reduced forms of a genus.[2] Between them they represent all positive integers not of the form[3] $\Lambda=7^{2k+1}(7m+3, 5, 6)$. The identities

$$\begin{aligned} x^2 + y^2 + 14z^2 &= x^2 + 2((y + 7z)/3)^2 + 7((y - 2z)/3)^2 \\ &= y^2 + 2((x + 7z)/3)^2 + 7((x - 2z)/3)^2 \end{aligned}$$

show that every number represented by g with either $y\equiv -z$ or $x\equiv -z \pmod 3$ is also represented by f. Hence every number $3n$ and $3n+1$ not of the form Λ is represented by f. For, $x\equiv y\equiv 0$, $z\not\equiv 0$, and x, $y\not\equiv 0$, $z\equiv 0 \pmod 3$ both imply $g\equiv 2$. If $N=3n$ or $3n+1$ is of the form Λ, then $7\mid N$, so that $N-13\cdot 3^2\neq\Lambda$. Similarly, one of $3n+2-13$ and $3n+2-52$ is not of the form Λ; but neither of these is congruent to 2 (mod 3). These linear forms are positive if $n\geqq 39$; h represents all integers not less than 119. The only number less than 119 not represented in (1, 2, 7, 13) is found to be 5.

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[1] This Bulletin, vol. 44 (1938), pp. 141–144.

[2] See any table of positive ternaries.

[3] For example, see B. W. Jones, Transactions of this Society, vol. 33 (1931), pp. 111–124.