ON THE COMPLETENESS OF A CERTAIN METRIC SPACE WITH AN APPLICATION TO BLASCHKE'S SELECTION THEOREM[1]

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1. **Introduction.** The purpose of this note is to prove that the metric space whose elements are the closed, bounded, non-null subsets of a complete metric space, and whose metric is the Hausdorff distance, is complete; and, using this result and others already known, to give a simple proof of Blaschke's selection theorem.

2. **Preliminaries.** Let K be a metric space with elements $x, y, \cdots$ and distance function $d(x, y)$. A sequence $x_1, x_2, \cdots$ in K such that $\sum_1^\infty d(x_i, x_{i+1})$ converges has been called an absolutely convergent sequence by MacNeille[2] [7, p. 192]. Every absolutely convergent sequence is a Cauchy sequence, and every Cauchy sequence contains absolutely convergent subsequences.

Let K^* be a metric space whose elements $X, Y, \cdots$ are the closed, bounded, and non-null subsets of K, and whose distance function $D(X, Y)$ is the Hausdorff distance between the sets X and Y (see Hausdorff [5, pp. 145–146] and Kuratowski [6, pp. 89–90]).

3. **The theorem.** *If K is complete, then K^* is also complete.*

Let $X_1, X_2, \cdots$ be any Cauchy sequence in K^*; without loss of generality we can assume that it is absolutely convergent. We shall define a set X and show that it is the limit of the given sequence. Let x_1 be any point in X_1, x_2 any point in X_2 such that $d(x_1, x_2) < D(X_1, X_2) + 2^{-1}$, x_3 any point in X_3 such that $d(x_2, x_3) < D(X_2, X_3) + 2^{-2}$, and so on. The existence of points $x_2, x_3, \cdots$ with the properties stated follows from the definition of the Hausdorff distance. Every point x_i in X_i is a member of a sequence $x_1, x_2, \cdots$ of the kind described. The sequence $x_1, x_2, \cdots$ is absolutely convergent and hence a Cauchy sequence; since K is complete, it has a limit x_0 in K. Let X_0 be the locus of all the points x_0 obtained as the limits of all possible sequences formed in the manner stated; let X be the closure of X_0. Then X is closed, bounded, and non-null, and X is in K^*. We shall show that $\lim X_k = X$. Let any $\epsilon > 0$ be given. Choose $n = n(\epsilon)$ so that $\sum_n^\infty [D(X_i, X_{i+1}) + 2^{-i}] < \epsilon/2$. Let $x^* \varepsilon X$, and let x_0 be the limit of a

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[2] Numbers in square brackets refer to the references at the end.