AN EXTENSION OF A COVARIANT DIFFERENTIATION PROCESS[1]

MARIE M. JOHNSON

Craig[2] has considered tensors $T^{\alpha\cdots}_{\beta\cdots}$ whose components are functions of n variables represented by x and their m derivatives $x', x'', \cdots, x^{(m)}$. He obtained the covariant derivative

$$(1) \qquad T^{\alpha\cdots}_{\beta\cdots x^{(m-1)\gamma}} - mT^{\alpha\cdots}_{\beta\cdots x^{(m)\lambda}}\left\{{}^{\lambda}_{\gamma}\right\}, \qquad m \geqq 2,$$

where

$$(2) \qquad \left\{{}^{\lambda}_{\gamma}\right\} \equiv x'^{\alpha}\Gamma^{\lambda}_{\gamma\alpha} + (1/2)x''^{\beta}f_{\gamma\delta\beta}f^{\delta\lambda},$$

and partial differentiation in (1) is denoted by the added subscript. Throughout, a repeated letter in one term indicates a sum of n terms. The purpose of this note is to derive another tensor from $T^{\alpha\cdots}_{\beta\cdots}$ whose covariant rank is one larger. The general process will be shown clearly by using $T^{\alpha}(x, x', x'', x''')$.

The extended point transformation

$$x^{\alpha} = x^{\alpha}(y), \qquad x'^{\alpha} = \frac{\partial x^{\alpha}}{\partial y^{i}} y'^{i},$$

$$x''^{\alpha} = \frac{\partial x^{\alpha}}{\partial y^{i}} y''^{i} + \frac{\partial^2 x^{\alpha}}{\partial y^{i}\partial y^{j}} y'^{i}y'^{j}, \cdots, \qquad \alpha = 1, \cdots, n,$$

gives the tensor equations of transformation of the tensor T^{α} as

$$(3) \qquad \overline{T}^{i}(y, y', y'', y''') = T^{\alpha}(x, x', x'', x''')\partial y^{i}/\partial x^{\alpha},$$

where y stands for the n variables $y^1, y^2, \cdots, y^n$ and a similar notation is used for the derivatives y', y'', and y'''. On differentiating equations (3) with respect to y'^{k} it is found that

$$(4) \qquad \overline{T}^{i}_{y'^{k}} = \left(T^{\alpha}_{x'^{\beta}}\frac{\partial x^{\beta}}{\partial y^{k}} + T^{\alpha}_{x''^{\beta}}\frac{\partial x''^{\beta}}{\partial y'^{k}} + T^{\alpha}_{x'''^{\beta}}\frac{\partial x'''^{\beta}}{\partial y'^{k}}\right)\partial y^{i}/\partial x^{\alpha}.$$

The derivatives can be expressed by using the following general formulas:

---

[1] Presented to the Society, April 15, 1939.

[2] H. V. Craig, *On a covariant differentiation process*, this Bulletin, vol. 37 (1931), pp. 731–734.