TYPICALLY-REAL FUNCTIONS WITH $a_n = 0$ FOR $n \equiv 0 \pmod 4$[1]

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1. **Introduction.** Let

$$f(z) = z + \sum_{2}^{\infty} a_n z^n \tag{1.1}$$

be typically-real for $|z| < 1$; that is, $f(z)$ within this circle is regular and takes on real values when and only when z is real. In particular, if $f(z)$ is univalent for $|z| < 1$ and has real coefficients, it is also typically-real. We suppose in addition that

$$a_n = 0 \qquad \text{for } n \equiv 0 \pmod 4). \tag{1.2}$$

In this paper we obtain sharp inequalities for the coefficients a_n.

Sharp inequalities for a_n are already well known[2] with the more restrictive condition

$$a_n = 0 \qquad \text{for } n \equiv 0 \pmod 2 \tag{1.3}$$

holding. In this case $|a_n| \leqq n$ with equality occurring for the odd function $(z+z^3)(1-z^2)^{-2}$. If besides, $f(z)$ is univalent and real on the real axis, the coefficients are bounded and satisfy[3] the inequalities

$$|a_{2n-1}| + |a_{2n+1}| \leqq 2, \qquad |a_3| \leqq 1. \tag{1.4}$$

With the less restrictive condition (1.2) replacing (1.3) the author obtains the following new and sharp inequalities:

$$|a_n| + 2^{-3/2}[(n-2)|a_{2m}| + n|a_2|] \leqq n, \qquad m, n \text{ odd}, n > 1; \tag{1.5}$$

$$|a_n| + 2^{-1/2}(n-1)|a_2| \leqq n, \qquad n \text{ odd}; \tag{1.6}$$

$$|a_n| + |a_2| \leqq 2^{3/2}, \qquad |a_2| \leqq 2^{1/2}, \qquad n \text{ even}. \tag{1.7}$$

In each case the equality sign holds for the typically-real function

$$z(1 - 2^{1/2}z + z^2)^{-1} = 2^{1/2}\sum_{1}^{\infty} \sin n\pi/4 \cdot z^n.$$

Since this function is also univalent for $|z| < 1$, the inequalities above

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[1] Presented to the Society, September 8, 1939.

[2] See W. Rogosinski, *Über positive harmonische Entwicklungen und typisch-reelle Potenzreihen*, Mathematische Zeitschrift, vol. 35 (1932), pp. 93–121.

[3] See J. Dieudonné, *Polynomes et fonctions bornées d'une variable complexe*, Annales de l'École Normale Supérieure, vol. 48 (1931), pp. 247–358.