A THEOREM ON SURFACES[1]

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It is a fact which is familiar[2] to projective differential geometers that the non-rectilinear asymptotic curves on an analytic non-developable ruled surface, not a quadric, in ordinary three-dimensional space belong to linear complexes if, and only if, the ruled surface belongs to a linear congruence, that is, if, and only if, the ruled surface has two distinct or coincident rectilinear directrices. It is furthermore known[3] that in this case the asymptotic curves on the ruled surface are projectively equivalent. It has also been demonstrated[4] that if the asymptotic curves on an analytic non-ruled surface S in ordinary space belong to linear complexes, then the asymptotic ruled surfaces of S, that is, the ruled surfaces composed of the tangents of the asymptotic curves of either family, constructed at the points of a fixed asymptotic curve of the other family, on S have rectilinear directrices and therefore are such that their asymptotic curves belong to linear complexes. The converse of this theorem is also true, as will be shown below. Moreover, it will be proved below, by the aid of some formulas computed[5] by MacQueen and the author, that in this case the asymptotic curves of one family on the surface S are projectively equivalent, as are also the asymptotic curves of the other family on S.

In attempting to determine whether the asymptotic curves on the asymptotic ruled surfaces of a non-ruled surface S are twisted cubics if the asymptotic curves on the surface S are twisted cubics, the author answered this question in the affirmative and discovered the following general theorem, which seems to have escaped notice hitherto and which it is the purpose of this note to put on record and demonstrate:

If the asymptotic curves on an analytic non-ruled surface S in ordinary space belong to linear complexes, then the asymptotic curves of each family on S are projectively equivalent, not only to each other, but also

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[1] Presented to the Society, December 2, 1939.

[2] A. Peter, *Die Flächen deren Haupttangentenkurven linearen Komplexen angehören*, Leipzig, dissertation, 1895.

[3] C. T. Sullivan, *Properties of surfaces whose asymptotic curves belong to linear complexes*, Transactions of this Society, vol. 15 (1914), pp. 167–196. See p. 171.

[4] Sullivan, loc. cit., p. 178; also Peter, loc. cit.

[5] E. P. Lane and M. L. MacQueen, *Surfaces whose asymptotic curves are twisted cubics*, American Journal of Mathematics, vol. 60 (1938), pp. 337–344. See p. 339.