NOTE ON THE CURVATURE OF ORTHOGONAL TRAJECTORIES OF LEVEL CURVES OF GREEN'S FUNCTION. III

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If R is a simply connected region of the extended (x, y)-plane with boundary B, and if Green's function $G(x, y)$ exists for R with pole in the finite point O, we denote by $\{T\}$ the set of orthogonal trajectories to the level curves $G(x, y) = \log r$, $0 < r < 1$, in R. The totality of circles each osculating at O one of the set of curves T passing through O consists precisely of the set of circles through O and through another fixed point D, depending on O and R. The point D is called the *conjugate of O with respect to R*. The term "circle" is here and below used in the extended sense, to include straight line, unless otherwise noted.

In a series of papers[1] the writer has recently studied some of the properties of the point D, notably (in M and I) that every circle through O and D cuts B; and (in II) that every point exterior to R is the conjugate with respect to R of a suitably chosen point O interior to R. It is the object of the present note to establish still further properties of the conjugate, namely the following theorems:

THEOREM 1. *Let R be a simply connected region of the w-plane with at least two boundary points. Let C be a circle intersecting the boundary of R in the finite point α. Let C be the boundary of a circular region R' (a half-plane, interior of a circle, or exterior of a circle, boundary points not included) whose points lie in R, and let T be a triangle contained in R', with the vertex α. Let the sequence of points $w_1, w_2, \cdots$ lie in T and approach α. Then the conjugate of w_n with respect to R also approaches α as n becomes infinite.*

THEOREM 2. *Let R be a simply connected region of the w-plane with at least two boundary points, and let w_0 be a boundary point of R. Then there exists a sequence of points $w_1, w_2, \cdots$ in R approaching w_0 such that the conjugate of w_n with respect to R approaches w_0.*

THEOREM 3. *There exists a limited Jordan region R of the w-plane, a boundary point w_0 of R, and a sequence $w_1, w_2, \cdots$ of points of R approaching w_0 such that the conjugate of w_n with respect to R becomes infinite with n.*

[1] American Mathematical Monthly, vol. 42 (1935), pp. 1–17; Proceedings of the National Academy of Sciences, vol. 23 (1937), pp. 166–169; this Bulletin, vol. 44 (1938), pp. 520–523. We shall refer to these papers as M, I, II respectively.