FACTORIZATION AND SIGNATURES OF LORENTZ MATRICES[1]

WALLACE GIVENS

The linear transformation $x \rightarrow \bar{x} = Lx$ (that is, $\bar{x}^i = L_j^i x^j$ with $i, j = 1, \cdots, n$) is called Lorentz if $(\bar{x}, \bar{x}) = (x, x)$, where

$$(1)\quad (x, x) = (x^1)^2 + (x^2)^2 + \cdots + (x^t)^2 - (x^{t+1})^2 - \cdots - (x^n)^2.$$

The matrices of all such transformations make up a group which we shall call the Lorentz group $\mathfrak{L}_{t,n-t}$. For $t=3$, $n=4$ it is usually called the extended Lorentz group.

In this paper we give extremely elementary proofs of two theorems. The first theorem has to do with the expression of a Lorentz matrix as a product of Lorentz matrices of simple type. For this it is sufficient that the elements of our matrices be chosen from a field of characteristic different from two. When the field is that of complex numbers, the signature of the quadratic form (1) is unimportant. The second theorem describes certain subgroups of the Lorentz group and for it we need an ordered (hence not a finite) field.

Each vector v for which $(v, v) \neq 0$ determines a transformation T_v with the equations

$$(2)\qquad \bar{x} = x - 2\frac{(v, x)}{(v, v)} v,$$

where

$$(3)\qquad (v, x) = v^1 x^1 + \cdots + v^t x^t - v^{t+1} x^{t+1} - \cdots - v^n x^n.$$

It is easy to verify the following:

(i) T_v is Lorentz.

(ii) The result of performing T_v twice is the identity.

(iii) Every multiple of v is a solution of the equations $\bar{x} = -x$, and conversely every solution is a multiple of v.

(iv) If $(v, y) = 0$ then $\bar{y} = y$ and conversely.

(v) For $v = e_j \equiv (0, \cdots, 0, 1, 0, \cdots, 0)$, where the one is in the jth place, T_{e_j} has the equations

$$(4)\quad \bar{x}^1 = x^1, \cdots, \bar{x}^{j-1} = x^{j-1}, \bar{x}^j = -x^j, \bar{x}^{j+1} = x^{j+1}, \cdots, \bar{x}^n = x^n.$$

(vi) If $v^1 = 0$, T_v has the equations

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