## COMBINATORIC INTERPRETATION OF A FORMULA FOR THE *n*TH DERIVATIVE OF A FUNCTION OF A FUNCTION

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Let  $f(x) = F(\phi(x))$ . The formula of Faa' di Bruno\* states that

(1) 
$$\frac{d^n f}{dx^n} = \sum_{k=1}^n \frac{d^k F}{d\phi^k} \sum_s A_{ks} \prod_r^{(s)} \left( \frac{d^k r \phi}{dx^{k_r}} \right)^{i_r},$$

where  $\prod_{r}^{(s)}$  is taken for a system (s) of positive integral solutions  $i_r$ ,  $k_r$  of the equations  $\sum_{r}i_r=k$ ,  $\sum_{r}k_ri_r=n$  and  $\sum_{s}$  is taken for all such systems. The factor  $A_{ks}$  is equal to  $n! [\prod_{r}i_r!(k_r!)^{i_r}]^{-1}$ . From a recent result of H. S. Wall† we have therefore that the numerical factor  $A_{ks}$  in (1) is equal to the number of ways that n different objects can be placed in  $k=\sum_{r}i_r$  compartments,  $k_r$  in each of  $i_r$  compartments, without regard to the order of arrangement of the compartments.

H. S. Wall expressed the *n*th derivative of f(x) in terms of logarithmic derivatives of f(x). Putting  $F(\phi) = e^{\phi}$ ,  $\phi(x) = \log f(x)$  in (1) his formula appears as a particular case of (1).

For functions of many variables  $f(x) = F(\phi_1(x), \phi_2(x), \cdots, \phi_n(x))$  the formula of F. G. Teixeira \( \) states that

$$\frac{d^n f}{dx^n} = \sum_{k=1}^n \sum_{\sigma} \frac{\partial^k F}{\prod_t^{(\sigma)} \partial x_t^{a_t}} \sum_{s} A_{ks\sigma} \prod_{tr}^{(s)} \left( \frac{d^{ktr} \phi_t}{dx_t^{ktr}} \right)^{i_{tr}},$$

where  $\prod_{t}^{(\sigma)}$  is taken for a system  $(\sigma)$  of nonnegative integral solutions  $a_t$  of the equation  $\sum_{t} a_t = k$  and  $\sum_{\sigma}$  is taken for all such systems;  $\prod_{t}^{(s)}$  is taken for a system (s) of positive integral solutions  $k_{tr}$ ,  $i_{tr}$  of the equations  $\sum_{r} i_{tr} = a_t$ ,  $\sum_{t} i_{tr} i_{tr} k_{tr} = n$  and  $\sum_{s}$  is taken for all such systems.  $A_{ks\sigma}$  is equal to  $n! \left[\prod_{t} i_{tr} i_{tr}! (k_{tr}!)^{i_{tr}}\right]^{-1}$ .  $A_{ks\sigma}$  has therefore the same combinatoric meaning as  $A_{ks}$ .

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<sup>\*</sup> F. Faa' di Bruno, Sullo sviluppo delle funzioni, Annali di Scienze Matematiche e Fisiche di Tortolini, vol. 6 (1855), pp. 479-480. For a bibliography on the subject see A. Voss, Encyclopädie der mathematischen Wissenschaften, II A 2, p. 88; E. Pascal, Esercizi Critici di Calcolo, Milano, 1921, 3d edition, pp. 111-112. See also L. S. Dederick, Annals of Mathematics, (2), vol. 27 (1926), pp. 385-394.

<sup>†</sup> H. S. Wall, On the nth derivative of f(x), this Bulletin, vol. 44 (1938), pp. 395-397; see the theorem p. 395, and formula (13).

<sup>‡</sup> Wall, loc. cit., formula (1).

<sup>§</sup> F. G. Teixeira, Sur les dérivées d'ordre quelconque, Giornale di Matematica di Battaglini, vol. 18 (1880), p. 306.