ON SERRET'S INTEGRAL FORMULA*

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In 1901 F. Morley† evaluated the integral

$$\int_0^{\pi/2} \phi^n \log^m (2 \cos \phi) d\phi \tag{1}$$

for even nonnegative integral values of n, and m a positive integer. His method was that of contour integration. However, the results could have been obtained with the help of a formula derived by A. R. Forsyth a few years earlier‡ and also from a formula derived by Cauchy as early as 1825.§ Cauchy's formula is

$$\int_0^{\pi/2} \cos^p \phi \cos q\phi d\phi = \frac{\pi}{2^{p+1}} \frac{\Gamma(p+1)}{\Gamma((p+q)/2+1)\Gamma((p-q)/2+1)}, \qquad R(p) > -1.$$

Here p and q are in general complex numbers with p subject to the restriction indicated. If this relation be written in the form

$$\int_0^{\pi/2} (2 \cos \phi)^p \cos q\phi d\phi = \frac{\pi}{2} \frac{\Gamma(p+1)}{\Gamma((p+q)/2+1)\Gamma((p-q)/2+1)}$$

and then differentiated m times with respect to p and n times (n even) with respect to q, and then p and q set equal to zero, the values of integrals of the form (1) are obtained quite easily.

In 1843 Serret|| obtained the formula

$$\begin{aligned}&\int_0^{\pi/2} \cos^p \phi \sin q\phi d\phi \\ &= \frac{\Gamma(p+1)}{\Gamma((p+q)/2+1)\Gamma((p-q)/2+1)} \int_0^1 \frac{t^{(p-q)/2} - t^{(p+q)/2}}{(1+t)^{p+1}(1-t)} dt, \qquad R(p) > -1.\end{aligned} \tag{2}$$

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† This Bulletin, vol. 7 (1901), p. 390.

‡ A. R. Forsyth, *Evaluation of two definite integrals*, Quarterly Journal of Mathematics, vol. 27 (1895), p. 221.

§ A. Cauchy, *Mémoire sur les Intégrales Définies Prises entre des Limites Imaginaires*, Paris, 1825, p. 40.

|| Alfred Serret, Journal de Mathématiques Pures et Appliquées, vol. 8 (1845), p. 7. The formula as given in Serret's article contains two misprints which are repeated in Encyklopädie der mathematischen Wissenschaften, vol. 2, part 2, no. A12, p. 853. They are corrected below.