THE DIRICHLET PROBLEM FOR THE VIBRATING STRING EQUATION

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This note considers the Dirichlet and Neumann type boundary value problem for the simple vibrating string equation. The detailed study for a special boundary is timely in view of certain categorical statements in the recent literature.* The results obtained below indicate how such statements are to be modified.† Of independent interest is the novel procedure, stemming from Lemma 1, for proving uniqueness in Theorems 1 and 2. The method is of wide utility and leads to interesting generalizations.

For convenience we use t for $v\tau$, where v and τ refer to the velocity of wave propagation and the time, respectively. The string equation is then

$$L[y] \equiv y_{xx} - y_{tt} = 0, \tag{1}$$

and the data are given on the boundaries of the finite rectangle

$$0 \leqq x \leqq S, \qquad 0 \leqq t \leqq T. \tag{1.1}$$

We denote the ratio T/S by α. The term "rectangle," used in the sequel, unless otherwise qualified, refers to the *closed* rectangle defined in (1.1).

We shall need the following lemma.

LEMMA 1. *If $y(x, t)$ is continuous in both real arguments in the rectangle, and if $F(p, u)$ is defined as*

$$F(p, u) = \int_0^T \int_0^S e^{i(xp+ut)} y(x, t)\,dx\,dt, \tag{2}$$

then $F(p, u)$ is an entire function in each of the complex variables p and u.

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* For example: "Dagegen würde ein Randwertproblem im Falle unserer hyperbolischen Differentialgleichung (our (1)) sinnlos sein," Courant-Hilbert, *Methoden der Mathematischen Physik*, vol. 2, p. 178. ". . . le problème de Dirichlet ne peut se poser pour le cas hyperbolique," J. Hadamard, L'Enseignement Mathématique, vol. 35 (1936), p. 26. Some of Hadamard's surmises are not borne out for the special situation we treat. Cf. pp. 26, 29 and notes on p. 29, loc. cit.

† In "principle," a physical realization of the Dirichlet problem is afforded by taking photographs of a vibrating string at two different times. However, "practically," our analysis is entirely ineffective, not for the reason of overdetermination (as in the illuminating instance on which Courant founds his mathematical conclusion) but because of the physical unpreciseness and inconstancy of the all important α.