A CHARACTERIZATION OF DEDEKIND STRUCTURES*

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If Σ is a Dedekind structure,† then for any two elements A and B of Σ, the quotient structures $[A, B]/A$ and $B/(A, B)$ are isomorphic. (Dedekind [**2**], Ore [**3**].) I prove here a converse result.

THEOREM. *Let Σ be a structure in which for every pair of elements A and B, the quotient structures $[A, B]/A$ and $B/(A, B)$ are isomorphic. Then if either the ascending or descending chain condition holds in Σ, the structure is Dedekindian.*

This result is comparatively trivial if *both* the ascending and descending chain conditions hold. That some sort of chain condition is necessary may be seen by a simple example. Consider a structure Σ with an all element O_0 and a unit element E_0 built up out of three ordered structures Σ_1, Σ_2, Σ_3 meeting only at O_0 and E_0, so that if $S_u \varepsilon \Sigma_u$, then

$$(S_u, S_v) = E_0, \qquad [S_u, S_v] = O_0$$

for $u, v = 1, 2, 3$, $u \neq v$. Then if each Σ_i is a series of the type of the real numbers in the closed interval 0, 1, the quotient structures of any pair $[S_u, S_v]/S_u$, $S_v/(S_u, S_v)$ are obviously isomorphic. But Σ is clearly non-Dedekindian.

The theorem is of some interest in view of the generalizations Ore has given of his decomposition theorems in Ore [**4**].

It suffices to prove the result under the hypothesis that the descending chain axiom holds in Σ (Ore [**3**, p. 410]). We formulate this axiom as follows:

(β) *If for any two elements A and B of Σ,*

$$A \supset X_1 \supset X_2 \supset X_3 \supset \cdots \supset B$$

for an infinity of X_i in Σ, all the X_i are equal from a certain point on.

Our proof rests upon several lemmas which we collect here.

LEMMA 1. (Dedekind [**2**].) *Σ is a Dedekind structure if and only if Σ contains no substructure Σ_0 of order five which is non-Dedekindian.*

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† We use the notation and terminology of Ore's fundamental paper, Ore [**3**], with the following two exceptions. (i) We write $A \supset B$, $B \subset A$ for Ore's $A \geqq B$, $B \leqq A$. (ii) If A is prime over B (Ore [**3**, p. 411]), we shall say "A covers B" or "B is covered by A" (Birkhoff [**1**]) and write $A > B$ or $B < A$.