ORTHOGONAL POLYNOMIALS WITH ORTHOGONAL DERIVATIVES*

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1. **Introduction.** Let $\{\phi_n(x) \equiv x^n + \cdots\}$ be a set of orthogonal polynomials satisfying the relations

$$
\begin{aligned}
&\int_a^b p(x)\phi_m(x)\phi_n(x)dx = \int_a^b q(x)\phi_m'(x)\phi_n'(x)dx = 0, \\
&\qquad\qquad\qquad\qquad m \neq n;\, m, n = 0, 1, \cdots, \\
&\alpha_i \equiv \int_a^b p(x)x^i dx, \qquad \beta_i \equiv \int_a^b q(x)x^i dx, \qquad i = 0, 1, \cdots, \\
&p(x) \geqq 0, \qquad q(x) \geqq 0, \qquad \alpha_0 > 0, \qquad \beta_0 > 0.
\end{aligned} \tag{1}
$$

Lebesgue integrals are used and the interval (a, b)† may be finite or infinite.

We are concerned with the following assertion:

THEOREM. *If $\{\phi_n(x)\}$ and $\{\phi_n'(x)\}$ are orthogonal systems of polynomials, then $\{\phi_n(x)\}$ may be reduced to the classical polynomials of Jacobi, Laguerre, or Hermite by means of a linear transformation on x.*

This result was first proved by W. Hahn‡ who obtained a differential equation of the second order for $\phi_n(x)$. When (a, b) is finite, Krall§ derived the Jacobi polynomials by using the moments β_i to determine the weight function $q(x)$. The present paper extends his method to the case (a, b) infinite, thus obtaining the Laguerre and Hermite polynomials.

2. **Weight function for $\{\phi_n'(x)\}$.** Krall's proof shows that constants r, s, t (not all zero) may be determined so that

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† There is no loss of generality in assuming the intervals of orthogonality for $\{\phi_n(x)\}$ and for $\{\phi_n'(x)\}$ to be the same, since the definitions of $p(x)$, $q(x)$ may always be extended to a common interval (a, b). More generally, $p(x)dx$ may be replaced by $d\psi_1(x) \equiv Ap(x) + dT(x)$, where A is a constant, and $\int_a^b x^i dT(x) = 0$, $(i = 0, 1, \cdots)$; $q(x)dx$ may be replaced by $d\psi_2(x)$, where $\psi_2(x)$ is monotone nondecreasing.

‡ W. Hahn, *Über die Jacobischen Polynome und zwei verwandte Polynomklassen*, Mathematische Zeitschrift, vol. 39 (1935), pp. 634–638.

§ H. Krall, *On derivatives of orthogonal polynomials*, this Bulletin, vol. 42 (1936), pp. 423–428.