EXPANSION OF FUNCTIONS IN SOLUTIONS OF FUNCTIONAL EQUATIONS*

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1. **Introduction.** In analysis a number of functional equations have solutions of the form

$$x^r \sum_{s=0}^{\infty} \alpha_{s,r} x^s. \tag{1}$$

Examples are (a) linear differential equations with a regular singular point at the origin, (b) the Volterra homogeneous integral equation with a regular singularity, (c) the linear q-difference equation, (d) the Fuchsian equation of infinite order. There are many others including mixed q-difference and differential equations.

Consider the equation

$$L(x, \lambda) \rightarrow y = 0 \tag{2}$$

where λ is a parameter and $L(x, \lambda)$ is an operator with the following property:

$$L(x, \lambda) \rightarrow x^p = x^p f(x, p, \lambda) = x^p \sum_{\mu=0}^{\infty} f_\mu(p, \lambda) x^\mu, \tag{3}$$

the series converging for $|x| \leqq N < r$ for all values of p, which may be a complex number. The purpose of this paper is to consider under what conditions a set of values $\{\lambda_m\}$, $(m=0, 1, 2, \cdots)$, can be determined so that for $\lambda = \lambda_m$ there will exist a solution of the form

$$\begin{aligned} y_{m+\sigma}(x) &= x^{m+\sigma} \sum_{s=0}^{\infty} \alpha_s^{(m+\sigma)} x^s = \sum_{s=0}^{\infty} \alpha_s^{(m+\sigma)} x^{m+\sigma+s} \\ &= x^{m+\sigma} \left\{ \alpha_0^{(m+\sigma)} h_m(x) \right\} \end{aligned} \tag{4}$$

such that an arbitrary function $x^\sigma f(x)$, $f(x)$ being analytic for $|x| < \rho$, can be expanded in a series

$$x^\sigma f(x) = \sum_{m=0}^{\infty} a_m y_{m+\sigma}(x) \tag{5}$$

which converges and represents the function in some region. For the

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