NON-CYCLIC ALGEBRAS WITH PURE MAXIMAL SUBFIELDS*

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One of the most elementary consequences of the assumption that a normal division algebra A is cyclic of degree n over its centrum K is that A contains a quantity j whose minimum equation is $\omega^n=\gamma$ in K. In 1933 I conjectured the truth of the converse proposition. The proof is easily reducible† to the case where n is a power p^e of a prime p. Let q be the characteristic of K. I succeeded in proving the theorem for $q=p$, e arbitrary,‡ as well as for $q\neq p$, $e=1$.§ There remained the case $q\neq p$, $e\geqq 2$.

My hope for the truth of the theorem was heightened by H. Hasse's remark|| that it would provide an essential simplification of the arithmetic existence properties required for the proof of the theorem that all normal division algebras over an algebraic number field are cyclic. However this hope is at an end. For the conjecture is actually false in the remaining case. This is shown by a demonstration of the validity of the following theorem:

THEOREM. *Let x, y, z be independent indeterminates over a field F of real numbers,¶ $K=F(x, y, z)$. Then there exist non-cyclic normal division algebras of degree and exponent four over K, each with a subfield $K(j)$ of degree four over K such that $j^4=\gamma$ in K.*

Our example is obtained from a class of non-cyclic algebras given in my paper in the Transactions of this Society, vol. 35 (1933), pp.

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† For, every A is a direct product of division algebras A_i of degrees $n_i=p_i^{e_i}$ for distinct primes p_i, and A is cyclic if and only if the A_i are cyclic. Moreover the field defined by $\omega^n=\gamma$ splits A if and only if the fields defined by $\omega^{n_i}=\gamma$ split the A_i. For references to the results used see M. Deuring's *Algebren*.

‡ Transactions of this Society, vol. 39 (1936), pp. 183–188.

§ Ibid., vol. 36 (1934), pp. 885–892.

|| In a letter to the author. The arithmetic existence theorem is that of W. Grünwald, Journal für die reine und angewandte Mathematik, vol. 169 (1933), pp. 103–107. The proof of Hasse applicable for the case $n=p$ is as follows. Assume that γ is in K and is exactly divisible by the first power of P for every prime ideal P of K such that the P-index of A is not unity. Let also γ be negative for all real fields conjugate to K. Then $K(\gamma^{1/n})$ splits A and is equivalent to a maximal subfield of A. When n is a prime p this implies that A is cyclic.

¶ Our existence theorem is for F non-modular. It seems likely that a modification can be made with F of characteristic any $p>2$.