THE GEOMETRY OF THE WHIRL-MOTION GROUP G_6: ELEMENTARY INVARIANTS*

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In this paper we shall study the elementary geometry of the oriented lineal elements of the plane with respect to the whirl-motion group G_6. We give results in addition to those found in a paper by Kasner, *The group of turns and slides and the geometry of turbines*, published in 1911 in the American Journal of Mathematics (vol. 33, pp. 193–202) and the paper by the authors, *The geometry of turbines, flat fields, and differential equations*, published in 1937 in the American Journal of Mathematics (vol. 59, pp. 545–563). The present paper can be read independently of the two earlier papers and contains the foundation of a new geometry of differential elements.

We begin by considering certain simple operations or transformations on the oriented lineal elements of the plane. A turn T_α converts each element into one having the same point and a direction making a fixed angle α with the original direction. By a slide S_k the line of the element remains the same and the point moves along the line a fixed distance k. These transformations together generate a continuous group of three parameters which we call the group of *whirl* transformations. It is noted that the group of whirls is isomorphic to the group of motions. These two three-parameter groups are commutative and together generate a new group of six parameters which we term the *whirl-motion group* G_6. It is our purpose to study the simple geometry of this group G_6. We find the fundamental invariants.

We define ∞^1 elements to be a *series* of elements; this includes a union or curve as a special case. A set of ∞^2 elements is said to form a *field* of elements, which corresponds to a differential equation of the first order $F(x, y, y')=0$. A *turbine* is the series which is obtained by applying a turn T_α to the elements of an oriented circle (which may be a null circle). A *flat field* is the field that is obtained from the totality of all elements which are determined by the set of all oriented circles containing a given element. We obtain the elementary relationships between elements, turbines, and flat fields.

For the analytic representation, it will be convenient to define an element by the coordinates (u, v, w) where v is the perpendicular from the origin, u is the angle between the perpendicular and the initial

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