ON THE nTH DERIVATIVE OF $f(x)$*

H. S. WALL

Let $y_1, y_2, y_3, \cdots$ be defined recursively as follows: y_1 is the logarithmic derivative of a function $y=f(x)$, and $y_\nu = D_x y_{\nu-1}$, $(\nu=2, 3, 4, \cdots)$. Then the successive derivatives $y', y'', y''', \cdots$ of y with respect to x are polynomials in y and the y_ν. In fact, $y'=yy_1$, $y''=y(y_2+y_1^2)$, $y'''=y(y_3+3y_1y_2+y_1^3)$, and

$$y^{(n)} = y \sum A^{(n)}_{\nu_1\nu_2\cdots\nu_n} y_1^{\nu_1} y_2^{\nu_2} \cdots y_n^{\nu_n}, \tag{1}$$

where $A^{(n)}_{\nu_1\nu_2\cdots\nu_n}$ is a positive integer and the summation is taken for all non-negative integral solutions $\nu_1, \nu_2, \nu_3, \cdots, \nu_n$ of the equation

$$\nu_1 + 2\nu_2 + 3\nu_3 + \cdots + n\nu_n = n. \tag{2}$$

This statement may readily be proved by mathematical induction. The principal object of the present note is to prove the following theorem:

THEOREM. *The integer $A^{(n)}_{\nu_1\nu_2\cdots\nu_n}$ in (1) is equal to the number of ways that n different objects can be placed in compartments, one in each of ν_1 compartments, two in each of ν_2 compartments, three in each of ν_3 compartments, $\cdots$, without regard to the order of arrangement of the compartments.*

1. Generalized binomial coefficients. Let k, m, n, $(kn \leqq m)$, be positive integers, and denote by $C^{(k)}_{m,n}$ the number of ways that kn objects can be selected from m objects and placed in n compartments, k in each compartment, where no account is taken of the order of arrangement of the compartments. Thus $C^{(1)}_{m,n}$ is the binomial coefficient $C_{m,n} = m!/[n!(m-n)!]$. We have

$$n!\cdot C^{(k)}_{m,n} = C_{m,kn}\cdot(C_{kn,k}\cdot C_{k(n-1),k}\cdot \cdots \cdot C_{k,k}),$$

or

$$C^{(k)}_{m,n} = m!/[n!(m-kn)!(k!)^n]. \tag{3}$$

This has meaning if $m \geqq kn$. For special 0 values of the indices we shall consider $C^{(k)}_{m,n}$ to be defined by (3) by taking $0!=1$. Thus if $k \geqq 0$, $m \geqq 0$, we have $C^{(k)}_{m,0}=1$.

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