A MULTIPLIER RULE IN ABSTRACT SPACES*

HERMAN H. GOLDSTINE

The Lagrange multiplier rule has been generalized to certain non-calculus of variations problems by Graves,† Hahn,‡ and the author.§ Moreover a very general problem was formulated by L. A. Lusternik.|| However his work seems to rest upon a theorem which is stated without proof and which the author is unable to verify. There are also certain other difficulties with his proof. The problem herein presented is so formulated that the problem of Bolza in the calculus of variations, the problems treated by Graves, Hahn, and the author, and the problem of minimizing a functional defined upon an arbitrary Banach space subject to very general numerically-valued side conditions, and numerous other examples are included as special cases.

The proof proceeds along lines which are essentially generalizations of the methods of the calculus of variations. This demonstration is made possible by the very powerful implicit function theorems of Hildebrandt and Graves¶ and yields analogs of the transversality condition and of the Euler-Lagrange equations. Some instances which explain the number of linear spaces involved are given in the concluding section.

1. **Definitions and assumptions.** In order to obtain our statement and proof of the multiplier rule it is convenient to refer to five normed linear spaces $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{X}$, $\mathfrak{U}$, $\mathfrak{V}$, of which $\mathfrak{M}$, $\mathfrak{U}$, and $\mathfrak{V}$ are assumed to be complete. It is further supposed that $\mathfrak{M}_0$, $\mathfrak{N}_0$, and $\mathfrak{X}_0$ are regions of their respective spaces. We shall then be concerned with the following basis:

* Presented to the Society, September 10, 1937.

† *A transformation of the problem of Lagrange in the calculus of variations*, Transactions of this Society, vol. 35 (1933), pp. 675–682.

‡ *Ueber die Lagrange'sche Multiplikatorenmethode*, Sitzungsberichte der Akademie Wien, vol. 131 (1922), pp. 531–550.

§ *The minima of functionals with associated side conditions*, Duke Mathematical Journal, vol. 3 (1937), pp. 418–425.

|| *Sur les extrêmes relatifs des fonctionnelles* (in Russian), Recueil Mathématique de la Société Mathématique de Moscou, vol. 41 (1934), pp. 390–401.

¶ See L. M. Graves, *Topics in the functional calculus*, this Bulletin, vol. 31 (1935), pp. 641–662; *Riemann integration and Taylor's theorem in general analysis*; *Implicit functions and differential equations in general analysis*; and T. H. Hildebrandt and L. M. Graves, *Implicit functions and their differentials in general analysis*. The last three articles cited appear in the Transactions of this Society, vol. 29 (1927), pp. 163–177, 514–552, and 127–153, respectively. In the sequel these papers will be referred to as papers I to IV, respectively.