ON SOME INEQUALITIES OF S. BERNSTEIN AND W. MARKOFF FOR DERIVATIVES OF POLYNOMIALS*

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A well known inequality on the derivatives of polynomials is that of S. Bernstein.†

BERNSTEIN'S THEOREM. *Suppose that $f(x)$ is a polynomial of degree n or less, and that in the interval $(-1, 1)$*

$$|f(x)| \leqq 1.$$

Then

$$|f'(x)|^2 \leqq \frac{n^2}{1-x^2}, \qquad -1 \leqq x \leqq 1,$$

and the equality can occur only if $f(x) \equiv \gamma T_n(x)$, $|\gamma| = 1$,‡ where $T_n(x)$ is the nth Tchebychef polynomial.

The extension of this theorem of Bernstein to the higher derivatives plays an important role in this paper. Thus, if $f(x)$ satisfies the conditions given in Bernstein's theorem, we obtain the inequality

$$|f^{(p)}(x)|^2 \leqq \left(\frac{d^p}{dx^p}\cos n\theta\right)^2 + \left(\frac{d^p}{dx^p}\sin n\theta\right)^2, \qquad x = \cos\theta,$$

for x in $(-1, 1)$. Using this inequality we are able to give a simple proof of W. Markoff's theorem,§ which states that under the conditions of Bernstein's theorem

$$|f^{(p)}(x)| \leqq \frac{n^2(n^2-1^2)(n^2-2^2)\cdots(n^2-(p-1)^2)}{1\cdot 3\cdot 5\cdot \cdots \cdot(2p-1)}, \quad -1 \leqq x \leqq 1.$$

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† S. Bernstein, *Sur l'ordre de la meilleure approximation des fonctions continues par des polynômes de degré donné*, Mémoires de l'Académie Royale de Belgique, (2), vol. 4 (1912), pp. 1–104. M. Riesz, *Eine trigonometrische Interpolationsformel und einige Ungleichungen für Polynome*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 23 (1914), pp. 354–368.

‡ γ stands hereafter for a constant, real or complex, of absolute magnitude 1.

§ W. Markoff, *Über Polynome, die in einem gegebenen Intervalle möglichst wenig von null abweichen*, Mathematische Annalen, vol. 77 (1916), pp. 213–258, translated by J. Grossman. The original appeared in Russian in 1892.

G. Szegö, *Über einen Satz von A. Markoff*, Mathematische Zeitschrift, vol. 23 (1925) pp. 45–61.