ABSTRACT RESIDUATION OVER LATTICES*

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Introduction. The idea of residuation goes back to Dedekind [3],† who introduced it in the theory of modules. It has since had extensive applications in the theory of algebraic modular systems [6], in the theory of ideals [8], and in certain topics of arithmetic [9]. On account of its fundamental role in several fields of modern algebra, it is desirable to consider residuation abstractly. A postulational treatment also is a necessary preliminary to the investigation of the structure properties of the residual. We give such an abstract formulation.

In a commutative ring with unit element the residual of an ideal B with respect to an ideal A, written $A:B$, is an ideal with the properties $A \supset (A:B)B$; if $A \supset XB$, then $A:B \supset X$. Although the residual is defined in terms of multiplication, most of its important properties are concerned with the cross-cut and union of ideals. Hence we shall consider a residual defined over a system having only these two operations, that is, over a *lattice* [2]. As an example of a system having a residual but no ordinary multiplication we consider in §5 residuation in a Boolean algebra.

In §1 the postulates for abstract residuation are given. Equality is taken as an undefined relation with cross-cut, union, and residual as undefined connections. In §2 we list a few systems satisfying the postulates. In §3 it is shown that the system defined by the postulates is a lattice and that the residual has all of its important properties which are independent of multiplication. Consistency and independence proofs are given in §4.

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1. **Postulates for residuation.** Let Σ be a set of elements A, B, C, $\cdots$; and let $=$, $[\ ,\ ]$, $(\ ,\ )$, and $:$ be relations, satisfying the postulates i–iv; 1–3; I–V. In what follows, $\circ$ denotes an arbitrary one of the relations $[\ ,\ ]$, $(\ ,\ )$, $:$ and the letters A, B, C, $\cdots$, appearing in the statement of the postulates indicate arbitrary elements of Σ.

POSTULATE i. *$A \circ B$ is in Σ whenever A and B are in Σ.*

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† Numbers in square brackets refer to the bibliography at the end of the paper.