THE RESOLVENT OF A CLOSED TRANSFORMATION

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1. **Introduction.** In reading over Chapter IV of Stone's book, *Linear Transformations in Hilbert Space*, I was impressed by the fact that a number of the results obtained are valid for any complex Banach space. This generality does not always appear at once evident, and it may be worth while to explain briefly.

The most interesting and important fact which underlies the material is that the resolvent of a closed distributive* transformation depends *analytically* on a parameter λ. This dependence is made precise in Stone's work with the aid of the inner product of Hilbert space; but this is not necessary, for it is known that the fundamental portions of the classical theory of analytic functions remain valid in complex Banach spaces.† In particular, Liouville's theorem admits a valid generalization. Thus we are able to prove that the spectrum of a (continuous) linear transformation whose domain is the whole space E is not empty. We shall now turn to the details.

2. **Preliminaries.** We use E to denote a complex Banach space; T will denote a distributive (additive, homogeneous) transformation, with domain and range both in E. We then write $T_\lambda = T - \lambda I$, and T_λ^{-1} will denote the inverse of T_λ when it exists. Here λ is a complex number, and I the identity transformation. We recall that a transformation admits an inverse if and only if it sets up a one to one correspondence between its domain and its range. When T is distributive, the necessary and sufficient condition that T_λ^{-1} exist is that $T_\lambda f = 0$ imply $f = 0$. The set of values of λ for which T_λ^{-1} is linear, with domain everywhere dense in E, is called the resolvent set of T. All other values of λ constitute the spectrum of T.

3. **Discussion of the resolvent.** We prove the following theorem:

THEOREM 1. *If $T_{\lambda_0}^{-1}$ exists and is linear, then T_λ^{-1} exists and is linear for each λ such that $|\lambda - \lambda_0| < 1/M_{\lambda_0}$, where M_{λ_0} is the modulus of $T_{\lambda_0}^{-1}$.*

* We use *distributive* where Stone uses *linear*, preferring to reserve the latter term for continuous distributive transformations. For the definition of a closed transformation see Stone, loc. cit., p. 38.

† For the independent variable a complex number this was pointed out by Wiener, Fundamenta Mathematicae, vol. 4 (1923). For the general case see A. E. Taylor, Comptes Rendus, vol. 203 (1936), pp. 1228–1230, and a forthcoming paper in the Annali della Reale Scuola Normale di Pisa; also L. M. Graves, this Bulletin, vol. 41 (1935), pp. 651–653.