ON BOUNDED CONVOLUTIONS

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For a type of convolutions of distributions on convex curves, there has been given recently a purely geometrical convergence criterion.† In what follows, this geometrical criterion will be freed from the convexity assumption, thus obtaining in case of bounded spectra a necessary and sufficient *convergence criterion which involves only the spectra* and not the distribution functions themselves. Only one-dimensional distributions will be considered; it will be clear from the proofs that the transition to multi-dimensional cases is obvious.

If $\beta=\beta(x)$ is a distribution function, let $S(\beta)$ denote its spectrum, and $[S(\beta)]$ the diameter of this spectrum, the diameter $[T]$ of a set T being defined as the least upper bound ($\leqq +\infty$) of the mutual distances of the pairs of points contained in T. The vectorial sum‡ of two sets T_1, T_2 will be denoted by $T_1(+)T_2$, so that $S(\beta_1{}^*\beta_2)$ is the closure of $S(\beta_1)\ (+)\ S(\beta_2)$; and $S(\beta_1{}^*\beta_2{}^*\cdots)$ is the closure of $S(\beta_1)(+)S(\beta_2)(+)\cdots$, whenever the infinite convolution $\beta_1{}^*\beta_2{}^*\cdots$ is convergent. An infinite convolution $\beta_1{}^*\beta_2{}^*\cdots$ will be called *bounded* if

$$\sum_{n=1}^{\infty}[S(\beta_n)]<+\infty. \tag{1}$$

It is clear that the boundedness of an infinite convolution is neither necessary nor sufficient for its convergence.

If $\beta_1{}^*\beta_2{}^*\cdots$ is convergent, its spectrum $S(\beta_1{}^*\beta_2{}^*\cdots)$ is a bounded set if and only if $\beta_1{}^*\beta_2{}^*\cdots$ is a bounded convolution in the sense (1). This is implied by the fact that *an infinite convolution $\beta_1{}^*\beta_2{}^*\cdots$ is bounded if and only if*

$$[S(\beta_1{}^*\cdots\beta_n{}^*)]<\text{const.}\qquad (n=1,2,\cdots). \tag{1'}$$

† E. R. van Kampen and A. Wintner, *Convolutions of distributions on convex curves and the Riemann zeta function*, American Journal of Mathematics, vol. 59 (1937), pp. 185–186.

‡ Cf. B. Jessen and A. Wintner, *Distribution functions and the Riemann zeta function*, Transactions of this Society, vol. 38 (1935), p. 52 and p. 56.