# NOTE ON THE RELATION BETWEEN CONTINUITY AND DEGREE OF POLYNOMIAL APPROXIMATION IN THE COMPLEX DOMAIN*

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1. *Introduction*. It is the purpose of the present note to establish the following theorems:

THEOREM I. *Let $C$ be an analytic Jordan curve in the $z$-plane and let $f(z)$ be defined in $\overline{C}$, the closed limited point set bounded by $C$. For each $n$, $n=1, 2, \cdots$, let a polynomial $P_n(z)$ of degree $n$ in $z$ exist such that*

$$(1) \qquad |f(z) - P_n(z)| \leqq \frac{M}{n^{p+\alpha}}, \qquad z \text{ in } \overline{C}, \qquad 0 < \alpha \leqq 1,$$

*where $M$ is a constant independent of $n$ and $z$, and $p$ is a non-negative integer. Then $f(z)$ is analytic in $C$ and continuous in $\overline{C}$; the $p$th derivative $f^{(p)}(z)$ exists on $C$ in the one-dimensional sense and satisfies the condition*

$$(2) \qquad |f^{(p)}(z_1) - f^{(p)}(z_2)| \leqq L|z_1 - z_2|^{\alpha} |\log |z_1 - z_2||^{\beta}, \qquad z_1, z_2 \text{ on } C,$$

*where $\beta = 0$ if $\alpha < 1$, and $\beta = 1$ if $\alpha = 1$, and where $L$ is a constant independent of $z_1$ and $z_2$.*

THEOREM II. *Let $E$, with boundary $C$, be a closed limited point set in the $z$-plane whose complement $K$ is connected, and is regular in the sense that there exists a function $w=\phi(z)$ which maps $K$ conformally but not necessarily uniformly onto $|w| > 1$ so that the points at infinity in the two planes correspond to each other. Let the locus $C_R$: $|\phi(z)| = R > 1$, consist of a finite number of mutually exterior analytic Jordan curves. Let $f(z)$ be defined in $E$, and for each $n$, $n=1, 2, \cdots$, let a polynomial $P_n(z)$ of degree $n$ in $z$ exist such that*

$$(3) \qquad |f(z) - P_n(z)| \leqq \frac{M}{n^{p+\alpha+1}R^n}, \qquad z \text{ in } E, \qquad 0 < \alpha \leqq 1,$$

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