ON SOME GAP THEOREMS FOR EULER'S METHOD OF SUMMATION OF SERIES

BY YOSHITOMO OKADA

Hardy and Littlewood* have proved the following theorem:

For a given series $\sum_{k=1}^{\infty} a_{n_k}$, $(a_{n_k} \neq 0)$, *let* θ *be a fixed constant such that*

$$\frac{n_{k+1}}{n_k} \geqq \theta > 1, \qquad (k = 1, 2, \cdots).$$

If this series be summable by Abel's method of summation to the sum s, *then this series is convergent and its sum is* s.

Obreschkoff† obtained also a similar result for Cesàro's method. We shall now study these results for Euler's method.

We shall begin with the following theorem:

THEOREM 1.‡ *Let* $\sum_{n=0}^{\infty} a_n$ *be a given series summable by Euler's method, that is, if* $s_0 = 0$, $s_n = a_0 + a_1 + \cdots + a_{n-1}$, $(n \geqq 1)$,

$$\lim_{n\to\infty} \frac{1}{2^n}\left\{ s_0 + ns_1 + \frac{n(n-1)}{2!} s_2 + \cdots + s_n \right\} = s \tag{1}$$

exists; and for two given increasing sequences $\{n_k\}$, $\{n_k'\}$, $(n_k < n_k')$, *of integers and for a given number* α, $(1 \leqq \alpha < 2)$, *let*

$$\begin{aligned} a_\nu &= 0, \text{ for } n_k < \nu < n_k', \qquad (k = 1, 2, \cdots), \\ a_n &= O(\alpha^n). \end{aligned} \tag{2}$$

If $\eta_k' / \eta_k \geqq (1+\eta)/(1-\eta)$, $(k = 1, 2, \cdots)$, *for a positive number* η *such that*

$$(1 + \eta) \log (1 + \eta) + (1 - \eta) \log (1 - \eta) - 2 \log \alpha > 0,$$

then

$$\lim_{k\to\infty} \sum_{\nu=0}^{n_k} a_\nu = s. \tag{3}$$

* Hardy and Littlewood, Proceedings of the London Mathematical Society, (2), vol. 25 (1926).

† Obreschkoff, Tôhoku Mathematical Journal, vol. 32 (1930).

‡ If, in this theorem, (1) holds uniformly and O of (2) is independent of z when each a_n is a function of z, then (3) also holds uniformly.