FORD ON ASYMPTOTIC DEVELOPMENTS

The Asymptotic Developments of Functions Defined by Maclaurin Series. By Walter B. Ford. University of Michigan Studies, Scientific Series, vol. XI. University of Michigan Press, Ann Arbor, 1936. viii+143 pp.

The subjects of mathematical monographs of the present day are all too often intelligible only to the initiates of their immediate fields. By and large the problems which are enunciable in few and simple terms, and which yet are neither classical nor beyond the reach of analysis, are rare. The problem to which the present book is devoted is such a one. It is, in brief, the following: A function of the complex variable being given through the medium of its Maclaurin series, say

$$f(z) = \sum_{n=0}^{\infty} c_n z^n,$$

to determine its nature and structure in the vicinity of the point $z=\infty$. This problem, though a relatively narrow one to be sure, is one of primary importance, not only in the general field of the theory of functions of a complex variable, but also, as the book at hand amply proves, in the theory of differential equations of the Fuchsian type.

The problem is one of very substantial and inherent difficulty, and much research must needs be done before it will have been solved with any show of completeness. The inroad upon it which is here presented has led to excellent results. In its main outlines the method of attack is the familiar one through the calculus of residues, and the analysis centers chiefly upon the evaluation or appraisal of contour integrals. It is basically assumed that the Maclaurin coefficients c_n depend upon n in the manner $c_n=g(n)$, where $g(w)$, regarded as a function of the complex variable w, fulfills certain conditions of analyticity and order of magnitude in appropriate regions of the w plane. As one would expect, the description of $f(z)$ in the region about $z=\infty$ in general calls for the use of asymptotic representations. Since the domains of validity of such representations are always restricted, the determination of both the domains and the representations valid within them is of necessity carried out. This is done in particular for the case in which c_n is any rational function of n, for the case of functions of so-called exponential type, those in which $c_n \sim h/\Gamma(n+p)$ with h and p any complex constants, and for the functions which are designated as of the Bessel type, those in which

$$c_n \sim \frac{h}{\Gamma(n+p_1)\Gamma(n+p_2)}\,.$$

As an illustration of the theory of the latter, the asymptotic forms of the Bessel functions themselves are obtained, incidentally without any use being made of their differential equation.

These general function theoretic deductions occupy the first seven chapters of the book. The material is excellently organized and is presented with all the clarity it permits. The earlier chapters are to be regarded as largely expository, though not entirely so, the principal source of material being the extensive re-