A MEAN VALUE THEOREM FOR POLYNOMIALS AND HARMONIC POLYNOMIALS

BY J. L. WALSH

1. *Introduction.* We define a polynomial in z of degree n as any function that can be expressed in the form $a_0z^n+a_1z^{n-1}+\cdots+a_n$; we do not require $a_0\neq 0$. With this definition the following theorems are valid, as is our purpose to show in the present note:

THEOREM 1. *If $f(z)$ is an analytic function of z for the value $z=z_0$, then we have*

$$\lim_{h\to 0}\frac{f(z_0+h)+f(z_0+\omega h)+\cdots+f(z_0+\omega^{N-1}h)-Nf(z_0)}{h^N} = \frac{f^{(N)}(z_0)}{(N-1)!}, \tag{1}$$

where ω denotes the number $e^{2\pi i/N}$.

A function $f(z)$ is said to have the *polygonal mean value property* or more simply *the mean value property* if for fixed N and for every z_0 the value of $f(z_0)$ is the mean of the values of $f(z)$ at the vertices of every regular polygon of N sides whose center is z_0.

THEOREM 2. *A necessary and sufficient condition that a function analytic for all values of z have the mean value property is that it be a polynomial of degree $N-1$.*

THEOREM 3. *A necessary and sufficient condition that a real function $f(z)\equiv u(x, y)$ continuous for all values of z $(=x+iy)$ have the mean value property is that it be a harmonic polynomial of degree $N-1$.*

A harmonic polynomial in x and y of degree $N-1$ is defined as the real part of a polynomial in z of degree $N-1$.

2. *Proof of Theorem* 1. In preparation for the proof of Theorem 1 we first formulate a following well known and easily proved lemma.