# ON DERIVATIVES OF ORTHOGONAL POLYNOMIALS*

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1. *Introduction.* A set $\phi_0(x)=1$, $\phi_1(x)$, $\phi_2(x)$ of polynomials of degrees $0, 1, 2, \cdots$, is called a set of orthogonal polynomials if they satisfy

$$\int_a^b \phi_m(x)\phi_n(x)d\psi(x) = 0, \qquad \int_a^b d\psi(x) > 0, \qquad (m \neq n),$$

where $\psi(x)$ is a non-decreasing function of bounded variation. There is no restriction in assuming the highest coefficient 1.

It has been shown by W. Hahn† that if the derivatives also form a set of orthogonal polynomials, then the original set were Jacobi, Hermite, or Laguerre polynomials. His method consisted in showing that the polynomials satisfy a differential equation of the type

$$(a + bx + cx^2)\phi_n'' + (d + ex)\phi_n' + \lambda_n\phi_n = 0.$$

From this it followed that the set were Jacobi, Hermite, or Laguerre polynomials.

Here we propose to give a new proof of this result, our point of view being to answer the question: What conditions on the weight function result from assuming that both $\{\phi_n(x)\}$ and $\{\phi_n'(x)\}$ are sets of orthogonal polynomials? However, we shall assume that $(a, b)$ is a finite interval and $d\psi(x)=p(x)dx$,‡ where the weight function is $L$-integrable.

2. *A Relation for the Weight Function* $q(x)$. Let the set $\{\phi_n'(x)\}$ be orthogonal in the interval $(c, d)$, infinite or not, with the weight function $q(x)$, that is,

$$\int_c^d q(x)\phi_n'(x)\phi_m'(x)dx = 0, \qquad (m \neq n).$$

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† W. Hahn, *Über die Jacobischen Polynome und zwei verwandte Polynomklassen*, Mathematische Zeitschrift, vol. 39 (1935), pp. 634–638.

‡ Professor Shohat informs me that he has discussed the general case $d\psi(x)$.