ON THE ANALOG FOR DIFFERENTIAL EQUATIONS OF THE HILBERT-NETTO THEOREM

BY H. W. RAUDENBUSH, JR.

The Hilbert-Netto theorem for polynomials* has an analog, obtained by J. F. Ritt,† for differential polynomials.‡ By a differential polynomial is meant a polynomial in a finite number of unknown functions $y_1, \cdots, y_n$ of the independent variable x and a certain number of their derivatives. To any finite set of differential polynomials $F_1, \cdots, F_r$ there corresponds a finite system of ordinary algebraic differential equations $F_1=0, \cdots, F_r=0$. It is supposed that the coefficients of the F_i are all elements of a field $\mathfrak{F}$ of differentiable functions of x which is closed with respect to differentiation. Let G be any differential polynomial with coefficients in $\mathfrak{F}$ such that $G=0$ has every solution§ of the system. If we denote by Σ the totality of differential polynomials that are linear combinations of the F_i and a certain number of their derivatives with differential polynomials with coefficients in $\mathfrak{F}$ as coefficients, then Ritt's theorem is that some power of G is in Σ.

We show in the following that this is as far in this direction as the analogy between the theories of polynomials and differential polynomials extends. For our result is that, contrary to the easy conjecture, it is not generally true that for a given system there exists a single positive integer ρ such that any differential polynomial G, defined as above for that system, has the power G^ρ in the corresponding Σ. We give an example of a system for which there exist G's with arbitrarily high powers not in Σ. This result shows, in particular, why the ideal theory of

* See B. L. van der Waerden, *Moderne Algebra*, vol. 2, p. 66.

† J. F. Ritt, *Differential Equations from the Algebraic Standpoint*, Colloquium Publications of this Society, vol. 14. (Cited as α.)

‡ J. F. Ritt, *Systems of algebraic differential equations*, Annals of Mathematics, (2), vol. 36 (1935), p. 293. The term "differential polynomial" is equivalent to the terms "differential form" or "form" used in α.

§ See α for a precise definition. For an abstract formulation, see H. W. Raudenbush, Jr., *Ideal theory and algebraic differential equations*, Transactions of this Society, vol. 36 (1934), pp. 361–368.