ON GENERALIZATIONS OF LENGTH AND AREA†

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1. *Introduction.* In the Lebesgue theory of integration, if G is a bounded Lebesgue measurable point set on the x axis and $f(x)$ is a non-negative bounded Lebesgue measurable function on G, then the plane set H, consisting of all points (x, y) such that x is on G and $0 \leqq y \leqq f(x)$, is Lebesgue plane measurable and $m^{(2)}H = \int_G f(x)dx$. This relation may be proved by showing first that if h is a positive number, then the set H_h: $[x \text{ in } G, 0 \leqq y \leqq h]$ is Lebesgue plane measurable and

$$m^{(2)}H_h = h\, mG, \tag{A}$$

that is, the "area" of H_h is its "base" times its "altitude".

C. Carathéodory, W. Gross, and others have defined linear measure for sets not necessarily on a line, plane measure for sets not necessarily on a plane, and in general p-dimensional measure for sets in q-dimensional space, which are generalizations of the notion of curve length, surface area, and so on. Fundamental and simple as it seems, the question whether the generalizations of length and area under these definitions preserve, as do Lebesgue's, the euclidean relation that area is the product of length by length, has received no attention. In this paper we discuss, without answering completely, the simplest phase of this question.

2. *Axioms on General Measure.* We first point out some facts concerning general measure.

By postulating the existence of a set function satisfying five axioms, Carathéodory [**2**]‡ developed a general theory of measure in which most of the theorems of the usual Lebesgue theory have an analog. Hahn [**4**], page 444, modified Carathéodory's fifth axiom, and by its use proved also the important relation that the inner measure of a set is the upper limit of the measures of its closed subsets. Hahn's modified axiom is as follows.

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‡ Numbers in brackets refer to the bibliography.