REDUCIBLE BOOLEAN FUNCTIONS

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In this note I establish a condition that a Boolean function of n variables, say f, be reducible to a product of two Boolean functions f_1 and f_2, where f involves variables not occurring in f_i; and, similarly, that f be reducible to f_1+f_2, or to $f_1 \circ f_2$, or to $f_1 \Delta f_2$.* These results are of interest in connection with the general theory of Boolean operations, since every Boolean operation can be regarded as a Boolean function.

In order to state my results briefly, I use the symbol $\oplus$, which stands ambiguously for any one of the four operations $\times$, $+$, $\circ$, Δ. Thus each of my theorems really comprises four theorems, which can be obtained from the given theorem by substituting first $\times$ for $\oplus$ throughout, then $+$, then $\circ$, and then Δ. The theorems now follow.

THEOREM 1. *If a Boolean function*

$$f(x_1, \cdots, x_n)$$

be given, then a necessary and sufficient condition that there exist a g and an h, so that

$$f(x_1, \cdots, x_n) = g(x_1, \cdots, x_p) \oplus h(x_q, \cdots, x_n), \qquad (q \leqq p+1),$$

is that

* The operation $a \circ b$ is defined by $a \circ b = ab' + a'b$; and the operation $a\Delta b$ is defined by $a\Delta b = ab + a'b'$. These operations, which are mutually dual, are associative and commutative, and satisfy the further laws: $(a \circ b)' = a \circ b' = a\Delta b$, $a \circ 1 = a\Delta 0 = a'$, $a \circ a = a\Delta a' = 0$, $a \circ 0 = a\Delta 1 = a$. For a further discussion, see Bernstein's paper, *Postulates for Boolean algebra involving the operation of complete disjunction*, to appear in the Annals of Mathematics. For a detailed treatment of $a \circ b$, see the two papers by M. H. Stone: *Postulates for Boolean algebra and generalized Boolean algebra*, American Journal of Mathematics, vol. 57 (1935), pp. 703–732, and *Subsumption of the theory of Boolean algebras under the theory of rings*, Proceedings of the National Academy of Sciences, vol. 21 (1935), pp. 103–105. Stone writes $a\Delta b$ for what I designate by $a \circ b$, and he does not discuss the relation which I denote by $a\Delta b$.