WIENER ON THE FOURIER INTEGRAL

The Fourier Integral and Certain of its Applications. By Norbert Wiener. Cambridge University Press, 1933. xi+201 pp.

An analyst of the older generation would probably understand the term Fourier integral to refer to the formula

$$f(x) = \frac{1}{\pi}\int_0^\infty du \int_{-\infty}^\infty f(t) \cos u(t-x)dt,$$

which is valid for a very restricted class of functions. He would not find this formula in Wiener's treatise, which is really devoted to the theory of Fourier transforms, and the corresponding reciprocity relations. The Fourier transform of $f(x)$ is

$$g(x) = (2\pi)^{-1/2}\int_{-\infty}^\infty f(t)e^{-ixt}dt \tag{1}$$

with suitable interpretation of the integral which ordinarily does not exist in the sense of Lebesgue. The reciprocity theorem states that conversely $f(x)$ is the Fourier transform of $g(-x)$ with, conceivably, a different interpretation of the integral.

Wiener is concerned almost exclusively with three classes of measurable functions $f(x)$, namely, L_1, L_2, and S. Here the first two symbols have their customary meaning, S is the class of functions of uniformly bounded mean square modulus. In the first case (1) is an ordinary Lebesgue integral, $g(x)$ is continuous and vanishes at infinity. In the second case the integrals are limits in the mean of order two, and the transform belongs to L_2 [Plancherel's theorem]. The transformation is unitary so that $f(x)$ and $g(x)$ have the same norm in L_2. In the third case, the integral of the formal transform serves the same needs.

The book is grouped into an Introduction and four chapters. Of these Chapter 1 uses ideas from a paper by the author on Plancherel's theorem (Journal of Mathematics and Physics, M.I.T., vol. 7 (1928)). Chapters 2 and 3 are an elaboration of portions of the author's Bôcher Prize memoir (Annals of Mathematics, (2), vol. 33 (1932)). The last chapter is based on another paper by the author (Acta Mathematica, vol. 55 (1930)). The Introduction [45 pp.] gives a rapid outline of the main theorems on Lebesgue and Riemann-Stieltjes integrals, orthogonal series, and the Riesz-Fischer theorem. We note in passing that Theorem X_{21} is false, but as the author neither proves nor uses it, no harm is done.

Chapter 1 [26 pp.] is devoted to Plancherel's theorem mentioned above. The proof given by the author (one of his own; he has given several others) is ultimately based upon the fact that the function $\exp(-x^2/2)$ is its own Fourier transform and on related properties of Hermitian polynomials. If $f(x)\epsilon L_2$, and

$$f(x) \sim \sum_{n=0}^\infty f_n\psi_n(x), \tag{2}$$