# A FACTORIZATION THEORY FOR POLYNOMIALS IN $x$ AND IN FUNCTIONS $e^{\alpha x}$

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1. *Introduction.* In this note we consider the problem of determining all representations of a function of the form

$$f(x) = \sum_{n=0}^{N} \Phi_n(x) e^{\alpha_n x}, \tag{1}$$

where the $\Phi$'s are polynomials and the $\alpha$'s are constants, as a product of functions of the same form. The case in which the $\Phi$'s are constants has been discussed by J. F. Ritt.* As would be expected, the solution of the general problem possesses some features that are rather different from those appearing in the special case.

It is assumed, of course, that no one of the $\Phi$'s is identically zero, and that if $N>0$, no two of the $\alpha$'s are equal. The case of chief interest is that in which $N>0$ and in which the $\Phi$'s have no common zero. The discussion will be confined to this case. We select those of the $\alpha$'s for which the real parts are least, and of the constants so selected (if there be more than one) we select the one for which the coefficient of $(-1)^{1/2}$ is least. Let the constant so selected be denoted by $\alpha_0$. We assume that $\alpha_0=0$. The class of functions of the form (1) satisfying the conditions stated in this paragraph will be called $C$.

If $f(x), f_1(x), \cdots, f_s(x)$ are all of the form (1), and if $f(x)=f_1(x) \cdots f_s(x)$, we shall say that $f(x)$ is divisible by each of the functions $f_i(x)$, and each of the latter functions will be called a factor of $f(x)$. A function which is divisible only by itself and by functions of the form $Ae^{\alpha x}$, where $A$ and $\alpha$ are constants, will be called irreducible.

2. *Reduction to a Problem Concerning Polynomials.* Monomial factors of $f(x)$ are, in a certain sense, trivial. Henceforth we consider only factors having at least two terms. These factors may be taken as belonging to the class $C$.

Suppose that the function

* J. F. Ritt, *A factorization theory for functions* $\sum_{i=1}^{n} a_i e^{\alpha_i x}$, Transactions of this Society, vol. 29 (1927), pp. 584–596.