ON THE GENERALIZED VANDERMONDE DETERMINANT AND SYMMETRIC FUNCTIONS

BY ARNOLD DRESDEN

1. *Introduction.* E. R. Heineman has published* a direct method for expressing an arbitrary symmetric polynomial in n variables $a_1, a_2, \cdots, a_n$ in terms of the elementary symmetric functions $E_1, E_2, \cdots, E_n$ of these variables. His method was based on a combination of a formula for the general Vandermonde determinant developed by him, with a theorem of Muir.† In the present note we develop in a very elementary way a simple formula for the quotient of a general Vandermonde determinant by the simple alternant, in terms of the "homogeneous product sums" of weight s.‡ These homogeneous product sums can be expressed explicitly in terms of the elementary symmetric functions.§ The formula which is obtained gives, therefore, by the use of Muir's theorem, another straightforward method for the calculation of an arbitrary symmetric polynomial.

2. *A Theorem on Determinants.* It is an exercise in elementary algebra to prove that the determinants

$$\begin{vmatrix} 1 & -b-c & bc \\ 1 & -c-a & ca \\ 1 & -a-b & ab \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

are equal. If, instead of evaluating each of the determinants, we try to transform one into the other by elementary transformations, we are led to the following generalization of this simple fact.

THEOREM 1. *If* $a_1, \cdots, a_n$ *are arbitrary complex numbers and* $p_{ki} = (-1)^k \sum^{(i)} a_1 \cdots a_k$, *the sum to be extended over all products* k *at a time of* $a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n$, *then*

* E. R. Heineman, *Generalized Vandermonde determinants*, Transactions of this Society, vol. 31 (1929), p. 464.

† Muir, *Theory of Determinants*, vol. 4, p. 151; see also Muir and Metzler, *A Treatise on the Theory of Determinants*, p. 344.

‡ See, for example, MacMahon, *Combinatory Analysis*, vol. 1, p. 3.

§ For example, as in MacMahon, loc. cit., p. 4.