A NOTE ON BILINEAR FORMS

BY R. E. A. C. PALEY†

1. *Introduction.* In this note I consider the proof of the following theorem due to M. Riesz.‡

THEOREM. *Let $M^*(\alpha, \gamma)$ denote the maximum for variable x of*

$$\frac{\left(\sum_{n=1}^{N} B_n \left| \sum_{m=1}^{M} A_{mn} x_m \right|^{1/\gamma}\right)^{\gamma}}{\left(\sum_{m=1}^{M} C_m \left| x_m \right|^{1/\alpha}\right)^{\alpha}}, \tag{1}$$

where the numbers A_{mn}, $(1 \leqq m \leqq M,\ 1 \leqq n \leqq N)$, are fixed and the numbers B_n, $(1 \leqq n \leqq N)$, and C_m, $(1 \leqq m \leqq M)$, are all positive. Then log M^* (α, γ) *is a convex function of the variables (α, γ) in the triangle*

$$0 \leqq \gamma \leqq \alpha \leqq 1$$

of the (α, γ) plane.

I show here that it is sufficient to prove the theorem in the case where $\gamma = \alpha$, $(0 \leqq \alpha \leqq 1)$. Suppose the theorem to have been established in this case.

2. *Proof of the Theorem.* Let d_n, $(1 \leqq n \leqq N)$, be a set of positive numbers for which

$$\sum_{n=1}^{N} B_n d_n = 1. \tag{2}$$

Then, if $\gamma \leqq \alpha$ the numerator of (1) is the maximum, for all sets of d_n satisfying (2), of the expression

$$\left(\sum_{n=1}^{N} B_n \left| \sum_{m=1}^{M} A_{mn} x_m \right|^{1/\alpha} d_n^{1-\gamma/\alpha}\right)^{\alpha}. \tag{3}$$

† With profound regret, the Editors note that the author of this paper died on April 3, 1933. This paper therefore appears posthumously, though the author had read the proofs of it.

‡ M. Riesz, *Sur les maxima des formes bilinéaires, et sur les fonctions linéaires*, Acta Mathematica, vol. 49 (1926), pp. 465–497.