ON ALGEBRAIC EQUATIONS HAVING ONLY REAL ROOTS*

BY W. E. ROTH

Given the algebraic equation

$$(1)\quad f_1(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0,\ a_n \neq 0,$$

whose roots x_i, $i=1, 2, \cdots, n$, are all real. Then there exist equations, $f_\lambda(x)=0$, also of degree n, whose roots are the real numbers x_i^λ, $(i=1, 2, \cdots, n; \lambda=\pm 1, \pm 2, \cdots)$. If

$$(2)\quad S_j = x_1^j + x_2^j + \cdots + x_n^j, \qquad (j = 0, \pm 1, \pm 2, \cdots),$$

then the determinants

$$\Delta_k^{(\lambda)} = \begin{vmatrix} S_0 & S_\lambda & \cdots S_{(k-1)\lambda} \\ S_\lambda & S_{2\lambda} & \cdots S_{k\lambda} \\ \cdot & \cdot & \cdot \\ S_{(k-1)\lambda} & S_{k\lambda} & \cdots S_{(2k-2)\lambda} \end{vmatrix}, \qquad (k = 2, 3, \cdots, n),$$

are all positive according to Borchardt's theorem† provided the roots of $f_\lambda(x)=0$ are distinct. On the other hand, if $f_1(x)=0$ has exactly μ $(\mu \leqq n)$ distinct roots, all real, then for odd values of λ, $f_\lambda(x)=0$ will have exactly the same number of distinct roots; and for even values of λ, exactly $\mu-\nu$ distinct roots, where ν is the number of distinct pairs of numerically equal roots of $f_1(x)=0$ which differ only in sign. Under these hypotheses, it is known that

$$\begin{aligned} \Delta_k^{(\lambda)} &> 0, & (k = 2, 3, \cdots, \mu), \\ &= 0, & (k = \mu+1, \mu+2, \cdots, n), \end{aligned}$$

if λ is an odd positive or negative integer, and that

$$\begin{aligned} \Delta_k^{(\lambda)} &> 0, & (k = 2, 3, \cdots, \mu-\nu), \\ &= 0, & (k = \mu-\nu+1, \mu-\nu+2, \cdots, n), \end{aligned}$$

* Presented to the Society, November 28, 1931.

† Borchardt, Journal de Mathématiques, vol. 12 (1847), p. 58; Werke, Berlin, (1888), p. 24.