THE CRITICAL POINTS OF FUNCTIONS AND THE CALCULUS OF VARIATIONS IN THE LARGE*

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1. *Introduction.* The conventional object in a paper on the calculus of variations is the investigation of the conditions under which a maximum or minimum† of a given integral occurs. Writers have accordingly done little with extremal segments that have contained more than one point conjugate to a given point. An extended theory is needed for several reasons.

One reason is that in applying the calculus of variations to that very general class of dynamical systems or differential equations which may be put in the form of the Euler equations, it is by no means a minimum or a maximum that is always sought. For example, if in the problem of two bodies we make use of the corresponding Jacobi principle of least action the ellipses which thereby appear as extremals always have pairs of conjugate points on them, and do not accordingly give a minimum to the integral relative to neighboring closed curves, so that no example of periodic motion would be found by a search for a minimum of the Jacobi integral. In general if one is looking for extremals joining two points, or periodic extremals deformable into a given closed curve, the a priori expectation, as justified by the results of this paper, in general problems, would seem to be that many more solutions would fail to give a minimum than would give a minimum. Even if the ultimate object is

* An address presented before the Society at the request of the program committee, April 6, 1928.

† For work on the absolute minimum see Bolza, *Vorlesungen über Variationsrechnung*, 1909, pp. 419–437, and Tonelli, *Fondamenti di Calcolo delle Variazioni*, vol. 2. Further references will be found in these works.