# ISOLATED SINGULAR POINTS OF HARMONIC FUNCTIONS*

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Bôcher has given several theorems relating to the nature of a harmonic function in the neighborhood of an isolated singular point.† E. Picard‡ has proved two of the theorems given by Bôcher in the paper mentioned above. In both papers the following fundamental theorem occurs.

*If a function $f(x, y)$ is continuous and harmonic everywhere in the interior of a closed plane region with the exception of an isolated point $P$ in the neighborhood of which $f(x, y)$ tends to plus infinity for every mode of approach to $P$, $f$ is of the form $c \log (1/r)+V$ in this neighborhood, where $c$ is a positive constant, $r$ the distance from $(x, y)$ to $P$ and $V$ is a function harmonic everywhere in the neighborhood of $P$, including $P$ itself.*

The analogous theorem for three-space is also proved in each paper.

Picard's proof for the plane makes use of complex variables. However, as Bôcher points out, it is desirable to have an independent proof in order to follow out Riemann's idea of basing the theory of complex variables on the theory of real harmonic functions in two variables. Bôcher's discussion is general and applies to harmonic functions in any number of variables. However, both Bôcher and Picard, in the case of the theorem cited for three-space, apply Green's Formula to a region bounded partly by the surface

$$f(x, y, z)=\text{constant}.$$

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* Presented to the Society, October 31, 1925. See also a paper by O. D. Kellogg, *On some theorems of Bôcher concerning isolated singular points of harmonic functions*, which is to appear in the next issue of the BULLETIN.

† This BULLETIN, vol. 9 (1902–3), p. 455 ff.

‡ BULLETIN DE LA SOCIÉTÉ DE FRANCE, vol. 52 (1924).