

TONELLI ON CALCULUS OF VARIATIONS

Fondamenti di Calcolo delle Variazioni. By Leonida Tonelli. Bologna, Zanichelli. Vol. 1, 7+406 pp., 1921, 55 Lire; vol. 2, 8+660 pp., 1923, 80 Lire.

In its classical period, the calculus of variations depended for many of its pivotal theorems upon the theory of differential equations. This debt was partly offset by the contributions to the theory of boundary value problems made by Mason, Richardson, et al., in which existence and oscillation theorems were obtained, via the theory of integral equations, from the calculus of variations. But it is an entirely different procedure which underlies Tonelli's work. For the line integral whose extreme values are sought, is here studied by means of its direct functional dependence upon the curve along which it is taken. This is a new departure, at least in so far as its systematic use throughout the theory is concerned. Nothing of the sort is found in the texts based upon the Weierstrass theory (Kneser, Bolza). And there is not more than a suggestion of it in the first volume of Hadamard's *Leçons*.

By putting into the foreground the dependence of the integral upon the entire curve, the author focuses attention upon the "integral" aspects of the theory, rather than upon its "differential" characters. We get a calculus of variations which is concerned primarily with neighborhoods of curves, rather than with those of points. Another striking feature of the new theory, and one intimately connected with what has already been mentioned, is the preponderant importance of the theory of absolute extrema, which now assumes first place while the theory of relative extrema, which heretofore claimed the lion's share, is derived from it. There is possible, of course, a difference of opinion with regard to the fundamental significance of this new theory. Whether independence from the theory of differential equations is a desideratum may well be queried. It is very likely true that, for a long time to come at least, the classical theory will supply the most ready method of becoming acquainted with the calculus of variations and its problems. It must be admitted also that the importance of Tonelli's method would be greatly enhanced if it should prove possible to extend his theory to more general functionals than those which occur in the calculus of variations. On the other hand, the advances in the theory which the treatment of the integral as a function of lines makes possible seem to secure a permanent place for it.

The task of the reviewer has been considerably lightened by an article contributed by the author to volume 31 of this BULLETIN (p. 163), in which he explained how the functional conception of the calculus of variations led to the use of semi-continuity and related ideas which play a role of central importance in the theory. Volume I, for which its author was awarded the 1923 gold medal of the Italian Society of Sciences, is devoted