ON A CERTAIN FUNCTIONAL CONDITION*

BY J. P. BALLANTINE

A mean, x_3, between two numbers, x_1 and x_2, is obtained by use of the formula

$$p_1f(x_1)+p_2f(x_2)=(p_1+p_2)f(x_3), \tag{1}$$

where p_1 and p_2 are arbitrary weights and $f(x)$ is any of several functions. If the function chosen is x itself, the resulting mean is the arithmetic mean; if $1/x$, the harmonic mean; if $\log x$, the geometric mean; if x^2 the mean-square. Since this terminology affords no hint for a generalization, we may as well call the general mean given by (1) the f-mean.

We have named all the means in common use. Why not, by use of the above generalization, extend the notion to, say, the "sine-mean"? This will probably not be done, principally because the proposed mean does not possess a certain useful property which is characteristic of all the ordinary means. This property is simply that multiplication of x_1 and x_2 by any constant results in the multiplication of x_3 by the same constant.

Let us study this property, and see what conditions it imposes on the function whose mean possesses it. We will replace (1) by the symmetrical equations

$$\begin{gathered} p_1f(x_1)+p_2f(x_2)+p_3f(x_3)=0 \\ p_1+p_2+p_3=0\,. \end{gathered} \tag{2}$$

The desired property is expressed by

$$p_1f(ax_1)+p_2f(ax_2)+p_3f(ax_3)=0\,, \tag{3}$$

where p_1, p_2, p_3, x_1, x_2, x_3 are any set of numbers satisfying (2), and a is any constant. We desire to find all functions $f(x)$ which

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* Presented to the Society, May 2, 1925.