TWO GENERAL FUNCTIONAL EQUATIONS*

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The object of this paper is to discuss the functional equations

(1) $f(x+y)+g(x-y)=h(x)k(y), \quad h(x)\not\equiv 0, \quad k(x)\not\equiv 0,$

and

(2) $F(x+y)G(x-y)=H(x)+K(y), \quad F(x)\not\equiv 0, \quad G(x)\not\equiv 0,$

in which x and y are independent variables and $f(x)$, $g(x)$, $h(x)$, $k(x)$, $F(x)$, $G(x)$, $H(x)$, $K(x)$ are functions to be determined. Special cases of equations (1) and (2) have been discussed in the literature. Some of the more familiar special cases are

$$h(x)\equiv k(x)\equiv f(x)\equiv \psi(x), \quad g(x)\equiv 0,$$
$$g(x)\equiv k(x)\equiv f(x), \quad h(x)\equiv 2f(x),$$
$$G(x)\equiv 1, \ H(x)\equiv K(x)\equiv F(x)\equiv \varphi(x),$$
$$G(x)\equiv F(x), \ H(x)\equiv F^2(x), \ K(x)\equiv F^2(x)-1,$$
$$G(x)\equiv F(x), \ H(x)\equiv F^2(x), \ K(x)\equiv -F^2(x).$$

In this paper no relationships are assumed between the functions in equation (1) or the functions in equation (2). Furthermore, no restrictions (such as continuity, differentiability, etc.) are imposed on the functions. The variables, x and y, are not assumed real nor must they necessarily be complex. The author shows that the functions of equations (1) and (2) are expressible in terms of the functions $\varphi(x)$ and $\psi(x)$ which satisfy the Cauchy equations

$$\varphi(x+y)=\varphi(x)+\varphi(y), \tag{3}$$

$$\psi(x+y)=\psi(x)\psi(y), \tag{4}$$

given above as special cases of (1) and (2). The results of the paper are of sufficient generality to permit immediate

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