A PROPERTY OF CONTINUITY*

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If ξ and η are two points of the interval (a, b) in which the function $f(x)$ is continuous, then the function takes on all values between $f(\xi)$ and $f(\eta)$ as x changes from ξ to η. This property, which we shall designate by (A), is common to all continuous functions, but it is possessed also by other functions. It was shown, for example, by Darboux† that all derived functions possess property (A), and it was pointed out by Lebesgue‡ that still other functions which are neither continuous nor derived have the property.

The present note is concerned with functions having property (A). The character of the discontinuities that such a function may have is shown. Additional conditions which are sufficient to insure that the functions be continuous or continuous and monotone follow. A theorem stating that when all the discontinuities of a function are of a certain kind it has property (A) is proved, and a function having property (A) and having its set of points of continuity and its set of points of discontinuity each everywhere-dense is constructed.

If c is a point of discontinuity of a function $f(x)$ having property (A), then in any interval about c the function takes on all values between its maximum§ and minimum at c. If $f(x)$ is unbounded, for example from above, in the neighborhood of c, the function could have no maximum at c, in which case, obviously, $f(x)$ would have, in any interval about c, all values greater than its minimum at c.

As an immediate consequence of the character of the discontinuities of a function having property (A), it follows that *such a function will be continuous unless the set of values it assumes an infinite number of times fills at least one interval.* The converse of this statement is not true. An examination

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† ANNALES DE L'ECOLE NORMALE (2), vol. 4 (1875), pp. 109, 110.

‡ *Leçons sur l'Intégration*, p. 92.

§ Hobson, *Theory of Functions of a Real Variable*, p. 234, for definition of maximum and minimum of a function at a point.